

Impossibility in Belief Merging

Amílcar Mata Díaz¹ and Ramón Pino Pérez²

¹ *Departamento de Medición y Evaluación
Facultad de Humanidades y Educación
Universidad de Los Andes
Mérida, Venezuela
amilcarmata@ula.ve*

² *Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes
Mérida, Venezuela
pino@ula.ve*

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Abstract. With the aim of studying social properties of belief merging and having a better understanding of impossibility, we extend in three ways the framework of logic-based merging introduced by Konieczny and Pino Pérez. First, at the level of representation of the information, we pass from belief bases to complex epistemic states. Second, the profiles are represented as functions of finite societies to the set of epistemic states (a sort of vectors) and not as multisets of epistemic states. Third, we extend the set of rational postulates in order to consider the epistemic versions of the classical postulates of Social Choice Theory: Standard Domain, Pareto Property, Independence of Irrelevant Alternatives and Absence of Dictator. These epistemic versions of social postulates are given, essentially, in terms of the finite propositional logic. We state some representation theorems for these operators. These extensions and representation theorems allow us to establish an epistemic and very general version of Arrow's Impossibility Theorem. One of the interesting features of our result, is that it holds for different representations of epistemic states; for instance conditionals, Ordinal Conditional functions and, of course, total preorders.

Keywords: Belief merging, epistemic states, Arrow's Impossibility Theorem

1 Introduction

Belief merging studies the methods leading to extract a coherent piece of information from many sources, which may be mutually contradictory. The applications in this domain go from decision making, passing by medical diagnosis, policy planning, to automatic integration of data. Understanding the theoretical model, its properties and limits is important in order to know in which domains these methods are valid and to develop future applications.

From the early works in belief merging [16,17,13,12,14] there has been a feeling of closeness between this framework and the Social Choice Theory [2,23]. In particular, the way in which concrete merging operators are constructed -via the representation theorem- evokes the methods for constructing social choice functions. Some works [20,5,8] have been done in which certain aspects of social choice are explored in belief merging, mainly impossibility [2,4] and strategy-proofness [10,21].

In some of the previous works, for instance in Chopra et al. [5], the results of impossibility do not hold and the operators are strategy-proofness. In others works, like in Everaere et al. [8], the behavior of some classes of merging operators is studied with respect to the manipulation. Therein is proved that some classes of merging operators are not strategy-proofness. In that work the impossibility problems are not addressed.

Facing these opposed behaviors of some classes of merging operators, our goal is to better understand what is happening in terms of general rational properties expressed in the logical language of belief merging. In particular, we are interested in knowing if there is a general result of impossibility in this setting, such as the Arrow's Impossibility Theorem in Social Choice Theory.

In order to accomplish our study, we extend in this work the framework of logic-based merging presented by Konieczny and Pino Pérez (KPP) [16,17,13,14,15] in three ways. First, the representation of information is more general; actually, we continue with the abstract view of epistemic states present in [19]. Second, the representation of profiles will be also more general; we adopt a functional view. Third, we enrich the set of rational postulates by introducing social postulates inspired on the classical Arrow's postulates in Social Choice Theory [2]. All these postulates are formulated in a logical setting.

Remember that in the works of Konieczny and Pino Pérez the information (the beliefs) is defined by sets of propositional sentences. It is important to notice that the KPP framework generalizes the seminal belief revision operators presented by Alchourrón, Gardenfors and Makinson [1,9,11] in which the beliefs (alias epistemic states) have also the same kind of representation, that is, a set of propositional sentences.

However, the necessity of considering more complex representations of epistemic states has been stated in the work of Darwiche and Pearl [7], in particular in order to have a good behavior with respect to the iteration of the process. We adopt here, essentially, the definition proposed by Benferhat et al. [3].

We have to note that Meyer [20] gave a merging model of more complex epistemic states for belief merging. In Meyer's work the epistemic states were defined by ranking functions over valuations. However, in this context, Meyer did not develop any study of the logical properties of these operators, and, therefore, does not give representation theorems for his operators. In this work we use an abstract definition of epistemic states that can be instantiated in different concrete representations. For instance, Ordinal Conditional Functions (ranking functions) proposed by Spohn [22]; some kind of conditionals like rational consequence relations proposed by Lehmann and Magidor [18]; the total preorders can be also concrete representations of epistemic states.

The notion of profile, first given as a multiset of belief bases, is extended in the current work. Actually, the multiset representation has implicitly the anonymity of the sources of information. Thus, this is not compatible with the existence of a dictator even not with the existence of a manipulator. In order to have a representation of the profiles which is more flexible and eventually compatible with the existence of dictators and more general than multisets, we adopt the functional view: a profile will be a function of a finite set of agents in the set of epistemic states. Our view generalizes the notion of profiles as lists or vectors explicit in [5,8].

In the current work, we present some postulates given in terms of the finite propositional logic (the "visible" part of epistemic states) which define some classes of merging operators of abstract epistemic states. We state some key representation theorems for these operators. In particular, we give the logical versions of classical postulates involved in the Arrow's Impossibility Theorem: Standard Domain, Pareto Property, Independence of Irrelevant Alternatives and Absence of Dictator. As a matter of fact, the postulate equivalent to Transitive Explanations will be always satisfied by our basic operators. We prove some relationships between these new postulates and the classical postulates of merging. Then we establish a general result of impossibility: The epistemic state merging operators satisfying standard domain and independence are, in fact, dictatorial operators.

It is worth to note that, due to the fact that our notion of epistemic states has rich instantiations, our general Impossibility Theorem has instantiations in rich representations of epistemic states as those mentioned before: Ordinal Conditional Functions, rational consequence relations and total pre-orders.

It is interesting to remark that our approach makes a strong use of the Standard Domain Postulate. However, this postulate is incompatible with the representation of epistemic states as propositional formulas. Actually, we will prove that such kind of representation is not rich enough in order to satisfy Standard Domain. This result strengthens the utility and the necessity of considering complex epistemic states.

Our Dictator notion is actually a weak notion of dictator: at the level of beliefs, the result of the merging entails the beliefs of the Dictator. It is for this notion that our Impossibility Theorem holds. But this theorem does not hold for a strong version of dictator, namely: that the result of the merging, at the level of beliefs, be *equivalent* to the beliefs of the Dictator. This will be seen with some examples of operators at work.

The Syntax Independence Postulate, (IC3), is not present in its complete form. Actually this postulate implies a very strong form of anonymity and therefore it is incompatible with a notion of dictator. In our setting, this postulate is slightly weakened in order to render it compatible with the notion of dictator.

Following the early works of Katsuno and Mendelzon in Belief Revision [11], in which all the pieces of information have the same nature³, in this work all the pieces of information are (complex) epistemic states. In particular, the information playing the role of integrity constraints is represented as an epistemic state too. In this way we generalize the epistemic states revision operators proposed by Benferhat et al. [3] (see also [19]).

We have to say that the Fairness Postulate, (IC4), considered in the KPP framework is not present in this work. The main reason for putting aside this postulate is its specificity. The sole known merging operators satisfying this postulate are the operators defined by distances (see [13,15]). Actually, in [13] is proved that some very natural operators satisfy (IC4) iff they are defined using a distance. There is, also, another important reason for withdrawing (IC4): many important classes of well behaved operators that we believe are worth being considered, do not satisfy (IC4). The following example illustrates this situation.

Example 1. Anne and Bob have to travel from point A to point B . There are four paths to perform the travel: w_1 , w_2 , w_3 and w_4 . Anne thinks that the best way to carry out the travel is w_1 . Actually she thinks that w_i is better than w_{i+1} for $i = 1, 2, 3$. Bob thinks that the better way to do the travel is w_4 and the others possibilities are equally good. A common consensual result in this situation is to take the path w_1 , the path that Anne prefers and which is not so bad for Bob. Whereas the path w_4 is very far from Anne's preferred path.

As a matter of fact, in order to have a better road map of the postulates here considered and their relationships, we consider concrete spaces, namely those where the epistemic states are total pre-orders over valuations, and we define some examples of operators for which the satisfaction of postulates is analyzed. One important thing that our examples reveal, is that neither the Standard Domain Postulate, nor Independence Postulate (nor even Unanimity) are necessary conditions in order to have dictatorial operators.

This work is organized as follows: Section 2 is devoted to defining the concepts used throughout the paper. Section 3 is devoted to giving the syntactical postulates and their semantical counterparts and then to establish the basic representation theorems. In Section 4 we introduce the new postulates coming from Social Choice and state a general Impossibility Theorem, the main result of the paper. In Section 5, we construct operators in a concrete class of epistemic states. Finally we make some concluding remarks in Section 6. The proofs are in A.

³ For them the old beliefs and the new beliefs are propositional formulas.

2 Preliminaries

Let A be a set. A binary relation \geq over A is a total preorder if it is total (therefore reflexive) and transitive. Let \geq be a total preorder over A . We define the strict relation $>$ and the indifference \simeq associated to \geq as follows: $a > b$ iff $a \geq b$ and $b \not\geq a$; $a \simeq b$ iff $a \geq b$ and $b \geq a$. It is clear that any linear order is a total preorder.

Let \geq be a total preorder over A and C be a subset of A . We say that c is a maximal element of C with respect to \geq if $c \in C$ and for all $x \in C$, $x \geq c$. The set of maximal elements of C will be denoted $\max(C, \geq)$. The maximal elements of the whole set A will be denoted $\max(\geq)$. We will denote by $\geq|_C$ the restriction of \geq to C . The set of total preorders over a set A will be denoted by $\mathbb{P}(A)$.

Let us introduce an important example of total preorder: the lexicographical combination of two total preorders. Let A be a nonempty set and let \geq_1 and \geq_2 be two total preorders over A . We define $\geq^{\text{lex}(\geq_1, \geq_2)}$ over A by putting:

$$a \geq^{\text{lex}(\geq_1, \geq_2)} b \Leftrightarrow \begin{cases} a >_1 b, \text{ or} \\ a \simeq_1 b \ \& \ a \geq_2 b \end{cases}$$

It is not hard to see that $\geq^{\text{lex}(\geq_1, \geq_2)}$ is a total preorder over A and that the following equality holds

$$\max(\geq^{\text{lex}(\geq_1, \geq_2)}) = \max(\max(\geq_1), \geq_2) \quad (1)$$

Moreover, it is also easy to show that if either \geq_1 or \geq_2 is a linear order then $\geq^{\text{lex}(\geq_1, \geq_2)}$ is also a linear order.

Sometimes it is useful to associate some ranking functions to the total preorders over a set A . These are functions $f : A \rightarrow \mathbb{R}$ determining the total preorder \geq over A by $a \geq b$ if, and only if $f(a) \geq f(b)$. Actually, we define the *canonical ranking function*, r_{\geq} , associated to a total preorder \geq over A by putting

$$r_{\geq}(a) = \max\{n \in \mathbb{N} : \exists a_0, a_1, \dots, a_n \in A; a_{i+1} > a_i \text{ and } a_n = a\}$$

Let \mathcal{P} be the set of propositional formulas built over a finite set \mathcal{P} of atomic propositions. \ast will denote the set of non contradictory formulas. \ast/\equiv will denote the set of non contradictory formulas modulo logical equivalence. Let $\mathcal{W}_{\mathcal{P}}$ be the set of valuations of formulas in $\mathcal{L}_{\mathcal{P}}$. If φ is a formula in $\mathcal{L}_{\mathcal{P}}$, we denote by $[[\varphi]]$ the set of its models, i.e. $[[\varphi]] = \{w \in \mathcal{W}_{\mathcal{P}} : w \models \varphi\}$. If φ_i is a formula in $\mathcal{L}_{\mathcal{P}}$, for each $i \in I$ (where I is a finite set of indexes), we denote by $\bigwedge_{i \in I} \varphi_i$ the conjunction of all the formulas φ_i with i in I . When I is the empty set, $\bigwedge_{i \in I} \varphi_i \equiv \top$. If M is a nonempty set of valuations, we denote by φ_M a formula such that $[[\varphi_M]] = M$.

In this work a *belief base* will be represented by a formula in $\mathcal{L}_{\mathcal{P}}$. It encodes the set of propositions believed by an agent. The intuitive meaning of a (complex) *epistemic state* is to have, in addition to a belief base, other information, eventually null. Many concrete representations of epistemic states have been proposed. The first one is, of course, that of the AGM framework [1,9], where an epistemic state is a logical theory. Darwiche and Pearl [6,7] represent epistemic states by total preorders over $\mathcal{W}_{\mathcal{P}}$. Spohn [22] uses Ordinal conditional functions. An abstract model to represent epistemic states was presented by Benferhat et al. [3]. We will adopt this abstract representation throughout this paper except in Section 5, where we consider the concrete representation of total preorders over valuations as a realization of the abstract model.

Definition 1 (Epistemic space). A triple $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ is called an *epistemic space* if \mathcal{E} is a nonempty set, $\mathcal{L}_{\mathcal{P}}$ is the set of formulas over a set of propositional variables \mathcal{P} and B is a function from \mathcal{E} into $\mathcal{L}_{\mathcal{P}}$, such that the image of B modulo logical equivalence is all the set $\mathcal{L}_{\mathcal{P}}^{\ast}/\equiv$.

The elements of \mathcal{E} are called *epistemic states*; B is called a *belief function*; for any E in \mathcal{E} , $B(E)$ is called the *entrenchment beliefs* (or belief base) of E . Notice that if I is a nonempty set of valuations, there is E such that $B(E) \equiv \varphi_I$, since the image of B (modulo logical equivalence) is all the set $\mathcal{L}_{\mathcal{P}}^*/\equiv$.

As an example of an epistemic space we can consider a finite propositional language $\mathcal{L}_{\mathcal{P}}$, and then let \mathcal{E} be the set of all total preorders over the interpretations, that is $\mathcal{E} = \mathbb{P}(\mathcal{W}_{\mathcal{P}})$, and let B be the function that maps an epistemic state \geq in a formula φ_{\geq} having as models the maximal elements of \geq .

In order to introduce the notion of *profile*, we consider an epistemic space $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ and a well ordered set $(S, <)$ the elements⁴ of which are called *agents*. Let $\mathcal{F}^*(S)$ be the set of all nonempty finite subsets of agents. Each set N in $\mathcal{F}^*(S)$ can be seen as a *finite society of agents*. We can suppose that if $N = \{i_1, i_2, \dots, i_n\}$ the elements are listed in increasing way, i.e. $i_k < i_m$ whenever $k < m$.

Given N in $\mathcal{F}^*(S)$, a partition of N is a finite family $\{N_1, N_2, \dots, N_k\}$ contained in $\mathcal{F}^*(S)$ of pairwise disjoint sets such that their union is N .

Definition 2 (Profile). Given N in $\mathcal{F}^*(S)$, an N -profile is a function $\Phi : N \rightarrow \mathcal{E}$. We think of $\Phi(i)$ as the epistemic state of the agent i , for each agent i in N .

Given a N -profile, $\Phi : N \rightarrow \mathcal{E}$, for each agent i in N , E_i will denote $\Phi(i)$. Thus, if $N = \{i_1, i_2, \dots, i_n\}$ is a finite society of agents, it can be seen as an ordered tuple: $\Phi = (E_{i_1}, E_{i_2}, \dots, E_{i_n})$. When N is a singleton, suppose $N = \{i\}$, by abuse, E_i will denote the N -profile $\Phi = (E_i)$. In that case we write i -profile instead of $\{i\}$ -profile. Given N in $\mathcal{F}^*(S)$, the set of N -profiles will be denoted \mathcal{E}^N . The set of all profiles of epistemic states will be denoted $\mathcal{P}(S, \mathcal{E})$, that is, $\mathcal{P}(S, \mathcal{E}) = \bigcup_{N \in \mathcal{F}^*(S)} \mathcal{E}^N$.

Let $N = \{i_1, i_2, \dots, i_n\}$ and $M = \{j_1, j_2, \dots, j_m\}$ be two finite societies of agents in S . Consider an N -profile

$\Phi = (E_{i_1}, E_{i_2}, \dots, E_{i_n})$ and an M -profile $\Psi = (E_{j_1}, E_{j_2}, \dots, E_{j_m})$. We say that Φ and Ψ are equivalent, denoted $\Phi \equiv \Psi$, if $n = m$ and $E_{i_k} = E_{j_k}$ for $k = 1, \dots, n$. If Φ and Ψ are not equivalent, we write $\Phi \not\equiv \Psi$. From this, for any pair of agents i, j in S , if we consider an i -profile E_i and j -profile E_j , then $E_i \equiv E_j$ iff, seen as epistemic states, we have $E_i = E_j$. Thus, by abuse and being clear from the context, we write respectively $E_i = E_j$ and $E_i \neq E_j$ instead of $E_i \equiv E_j$ and $E_i \not\equiv E_j$. If N and M are disjoint we define a new $(N \cup M)$ -profile, the joint of Φ and Ψ , denoted $\Phi \sqcup \Psi$, in the following way:

$$(\Phi \sqcup \Psi)(i) = \begin{cases} \Phi(i), & \text{if } i \in N \\ \Psi(i), & \text{if } i \in M \end{cases}$$

If $M \subseteq N$, and M is nonempty, and Φ is an N -profile, then $\Phi \upharpoonright_M$ will denote the M -profile obtained by the restriction of Φ to M , that is, $\Phi \upharpoonright_M : M \rightarrow \mathcal{E}$, where $\Phi \upharpoonright_M(j) = \Phi(j)$, for each j in M .

Thus, if $\{N_1, N_2, \dots, N_k\}$ is a partition of a finite society N of agents in S , for each N -profile Φ , we have

$$\Phi = \Phi \upharpoonright_{N_1} \sqcup \dots \sqcup \Phi \upharpoonright_{N_k}$$

From now on, we will suppose that N is the finite society $\{i_1, i_2, \dots, i_n\}$, while the N -profiles Φ and Φ' will be denoted by $(E_{i_1}, E_{i_2}, \dots, E_{i_n})$ and $(E'_{i_1}, E'_{i_2}, \dots, E'_{i_n})$ respectively.

3 Epistemic states fusion operators

3.1 Postulates

We fix an epistemic space $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ and a set of agents S . A function of the form $\nabla : \mathcal{P}(S, \mathcal{E}) \times \mathcal{E} \rightarrow \mathcal{E}$ will be called an *epistemic state combination operator*, for short an ES combination operator. $\nabla(\Phi, E)$ represents the result of combining the epistemic states in Φ under the *integrity constraint* E .

⁴ The set $(S, <)$ can be identified with the set of natural numbers \mathbb{N} with the usual order.

Now we establish the rationality postulates of fusion in the setting of epistemic states. Most of them are adapted from IC merging postulates proposed by Konieczny and Pino Pérez [17,13,14] (a first version of these postulates appeared in [19]). Since the beliefs of epistemic states constitute their only aspect with well known logical structure, we express the rationality postulates in logical terms at the level of beliefs. Some of these postulates are new mainly due to the new presentation of profiles as functions of finite societies into epistemic states.

In order to introduce such postulates, let N and M be any pair of finite societies of agents in \mathcal{S} , Φ be any N -profile, Φ' be any M -profile, E, E', E'' be any triple of epistemic states.

(ESF1) $B(\nabla(\Phi, E)) \vdash B(E)$.

(ESF2) If $\Phi \equiv \Phi'$ and $B(E) \equiv B(E')$ then $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E'))$

(ESF3) If $B(E) \equiv B(E') \wedge B(E'')$ then $B(\nabla(\Phi, E')) \wedge B(E'') \vdash B(\nabla(\Phi, E))$.

(ESF4) If $B(E) \equiv B(E') \wedge B(E'')$ and $B(\nabla(\Phi, E')) \wedge B(E'') \not\vdash \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi, E')) \wedge B(E'')$.

(ESF1) tells us that the belief of the result has to be logically stronger than the belief of the restriction. This Postulate corresponds to Postulate (IC1) of IC merging operators [13,15].

(ESF2) is a weak form of the *anonymity* at the profile level and a *syntax irrelevance property* at the level of beliefs for the integrity restrictions. This Postulate corresponds⁵ to Postulate (IC3) of IC merging operators [13,15].

(ESF3) and **(ESF4)** together determine one important property in which the beliefs are chosen. They correspond to postulates (IC7) and (IC8) of IC merging operators respectively [13,15].

These four postulates, **(ESF1)**-**(ESF4)**, are called *basic epistemic state fusion postulates*. They are considered the minimal requirements of rationality that the combination operators have to satisfy.

Definition 3 (Basic fusion operators). Let $(\mathcal{E}, B, \mathcal{L}_\Phi)$ be an epistemic space and let \mathcal{S} be a set of agents. A combination operator of epistemic states ∇ is said to be an *epistemic state basic fusion operator* (ES basic fusion operator for short) if it satisfies **(ESF1)**-**(ESF4)**.

There are other important postulates describing mainly the relationships between the results of merging a whole society and the results of merging its subsocieties. In order to establish such properties, let j, k be a pair of agents in \mathcal{S} , N be any finite society of agents in \mathcal{S} , $\{N_1, N_2\}$ be any partition of N , Φ be any N -profile, E_j be any j -profile, E_k be any k -profile, and E be any epistemic state in \mathcal{E} .

(ESF5) If $E_j \neq E_k$, there exists E' in \mathcal{E} such that $B(\nabla(E_j, E')) \neq B(\nabla(E_k, E'))$

(ESF6) If $\bigwedge_{i \in N} B(E_i) \wedge B(E) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \equiv \bigwedge_{i \in N} B(E_i) \wedge B(E)$

(ESF7) $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E)) \vdash B(\nabla(\Phi, E))$

(ESF8) If $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E)) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$

(ESF5) is a new postulate. It says that given two different epistemic states, there is a restriction E that leads to different results at the level of beliefs, *i.e.* the beliefs of the result of the operator applied to each epistemic state with the restriction E , will not be equivalent.

(ESF6) expresses that if all the agents of the profile agree at the level of beliefs with the restriction, this agreement will coincide with the belief resulting after application of the operator. This Postulate corresponds to Postulate (IC2) of IC merging operators [13,15].

⁵ Actually, the version of equivalence with the profiles as multisets would correspond in our setting to allowing permutations in the profiles, but indeed we do not authorize it.

(ESF7) tells us that for any partition of a group into two subgroups, the conjunction of belief of the result of applying the operator to each subgroup will be logically stronger than the beliefs resulting of applying the operator to the whole group. This Postulate corresponds to Postulate (IC5) of IC merging operators [13,15].

(ESF8) expresses that if we can divide a group into two subgroups such that the application of the operator to each subgroup leads to beliefs which are mutually consistent, then the conjunction of these beliefs will be the beliefs resulting of applying the operator to the whole group. This Postulate corresponds to Postulate (IC6) of IC merging operators [13,15].

Definition 4 (Epistemic state fusion operators). Let ∇ be an ES combination operator. ∇ is said to be an epistemic state fusion operator (ES fusion operator for short) if it satisfies the postulates (ESF1)–(ESF8).

There are some important variants of these operators coming from the satisfaction of some special postulates. These postulates are stated in what follows:

(ESF8W) If $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E)) \not\models \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi \upharpoonright_{N_1}, E)) \vee B(\nabla(\Phi \upharpoonright_{N_2}, E))$

This property tells us that if a group is divided into two subgroups and, after application of the operator, the beliefs of the subgroups are consistent, then the beliefs of the whole group after application of the operator have to entail the disjunction of the beliefs of each subgroup. This Postulate corresponds to Postulate (IC6') of IC merging operators [13,15].

Replacing the postulate (ESF8) by the weaker postulate (ESF8W), gives us the class of epistemic state quasifusion operators.

Definition 5 (Epistemic state quasifusion operators). Let ∇ be an ES combination operator. ∇ is said to be an epistemic state quasifusion operator (ES quasifusion operator for short) if it satisfies the postulates (ESF1)–(ESF7) and (ESF8W).

Note that if an operator satisfies (ESF8), necessarily it satisfies (ESF8W). Thus, every ES fusion operator is an ES quasifusion operator. However, the converse is not true (see Section 5).

3.2 Faithful assignments

An *assignment* is a function mapping epistemic state profiles into total preorders over interpretations. The intended meaning of these mappings is encoding semantically, in some sense, the group preference.

Definition 6 (Assignment and Basic Assignment). Let $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ be an epistemic space and \mathcal{S} be a set of agents. An assignment is a function $f : \mathcal{P}(\mathcal{S}, \mathcal{E}) \rightarrow \mathbb{P}(\mathcal{W}_{\mathcal{P}})$ mapping each epistemic profile Φ into $f(\Phi)$, a total preorder over $\mathcal{W}_{\mathcal{P}}$. An assignment is called basic assignment when, for any profiles Φ and Ψ , if $\Phi \equiv \Psi$ then $f(\Phi) = f(\Psi)$.

Given an assignment f , we will denote by \geq_{Φ} , the total preorder $f(\Phi)$; and $\Phi \mapsto \geq_{\Phi}$ will denote the mapping f . The total preorder \geq_{Φ} can be seen as the group plausibility preference over worlds:

- If $w \geq_{\Phi} w'$, we will say that w is *at least as plausible as* w' , for the agents group in Φ
- If $w >_{\Phi} w'$, we will say that w is *more plausible than* w' , for the agents group in Φ

In order to have socially well behaved assignments, it is necessary to impose some rational properties. With the purpose of doing this, let j, k be a pair of agents in \mathcal{S} , N be any finite society of agents in \mathcal{S} , $\{N_1, N_2\}$ be any partition of N , Φ be any N -profile, E_j be any j -profile, E_k be any k -profile, and w, w' be any pair of interpretations in $\mathcal{W}_{\mathcal{P}}$.

- 1 If $E_j \neq E_k$, then $\succeq_{E_j} \neq \succeq_{E_k}$
- 2 If $\bigwedge_{i \in N} B(E_i) \not\models \perp$, then $[[\bigwedge_{i \in N} B(E_i)]] = \max(\succeq_\Phi)$
- 3 If $w \succeq_{\Phi \upharpoonright_{N_1}} w'$ and $w \succeq_{\Phi \upharpoonright_{N_2}} w'$ then $w \succeq_\Phi w'$
- 4 If $w \succeq_{\Phi \upharpoonright_{N_1}} w'$ and $w >_{\Phi \upharpoonright_{N_2}} w'$, then $w >_\Phi w'$

Property 1 imposes that different epistemic states lead to different total preorders (injectivity of the assignment restricted to profile of size one).

Property 2 tells us that, if there are models of the conjunction of the beliefs of the epistemic states of the profile, they are exactly the maximal models of the total preorder associated to the profile.

Property 3 expresses that if one model w_1 is at least as plausible as w_2 for one group, and the same occurs for a second group, then for the group resulting of the union of these groups, w_1 will be at least as plausible as w_2 .

Property 4 is similar to the previous one, except that if there is one preference strict for one of the subgroups, this will be the case for the whole group.

We think that the most entrenched preferences should represent the beliefs. This is expressed more precisely by the following property:

Definition 7 (Maximality Condition). Let $(\mathcal{E}, B, \mathcal{L}_\Phi)$ be an epistemic space and S be a set of agents. The assignment $\Phi \mapsto \succeq_\Phi$ satisfies the maximality condition with respect to B , if for all i in S and all i -profile E_i , the following equation holds

$$[[B(E_i)]] = \max(\succeq_{E_i})$$

Note that any assignment $\Phi \mapsto \succeq_\Phi$ satisfying Property 2, with respect to B , satisfies the Maximality Condition. Moreover, in presence of Properties 3 and 4, we have that Property 2 is equivalent to the Maximality Condition.

Proposition 1. Let $(\mathcal{E}, B, \mathcal{L}_\Phi)$ be an epistemic space and S be a set of agents. Suppose that the assignment $\Phi \mapsto \succeq_\Phi$ satisfies Properties 3 and 4. Then $\Phi \mapsto \succeq_\Phi$ satisfies Property 2 if, and only if, it satisfies the Maximality Condition.

The previous result shows some hidden connections between Properties 2, 3 and 4 under the Maximality Condition. Actually, under Maximality Condition, Properties 3 and 4 entail Property 2. However the converse does not hold as we will observe in Section 5.

The properties previously stated, together determine when an assignment has a good social behavior.

Definition 8 (Faithful assignment). Let $(\mathcal{E}, B, \mathcal{L}_\Phi)$ be an epistemic space and S be a set of agents. An assignment $\Phi \mapsto \succeq_\Phi$ is called a faithful assignment, with respect to B , if it is a basic assignment which satisfies Properties 1-4.

The following property is weaker than Property 4:

- 4' If $w >_{\Phi \upharpoonright_{N_1}} w'$ and $w >_{\Phi \upharpoonright_{N_2}} w'$ then $w >_\Phi w'$

This expresses that given two alternatives, w , w' , if for one of the groups, w is more plausible than w' , and the same relation occurs for other group, then, putting the groups together, w is still more plausible than w' .

Note that Property 4 entails Property 4'. However, the converse is not true as we will see in Section 5.

Replacing Property 4 by Property 4' in the definition of faithful assignments gives us another important class of assignments:

Definition 9 (Quasifaithful assignment). Let $(\mathcal{E}, B, \mathcal{L}_\Phi)$ be an epistemic space and S be a set of agents. An assignment $\Phi \mapsto \succeq_\Phi$ is called a quasifaithful assignment, with respect to B , if it is a basic assignment in which Properties 1-3 and 4' hold.

3.3 Representation theorems

We present some results which help to understand the behavior of the operators defined previously. This allows us to describe the operators, at least partially, in a semantical way. Thus, we call these results weak representation by opposition to some results of representation in concrete structures like in [19], where we can effectively build the operator starting from an assignment.

Theorem 1 (Weak representation for ES basic fusion operators).

An ES combination operator ∇ is an ES basic fusion operator iff there exists a unique basic assignment $\Phi \mapsto \geq_\Phi$ such that:

$$\llbracket B(\nabla(\Phi, E)) \rrbracket = \max(\llbracket B(E) \rrbracket, \geq_\Phi) \quad (\text{B-Rep})$$

It is worth to note that we can obtain a variant of this result with a weak version of **(ESF2)** where the equivalence between profiles is not necessary, just like the version presented in [19]. More precisely, the operators satisfying this weak version of **(ESF2)** plus **(ESF1)**, **(ESF3)** and **(ESF4)**, are exactly the operators weakly represented by a simple assignment.

The Weak representation theorem allows us to obtain very tight relations between the (syntactical) postulates for fusion and the properties of the assignments. More precisely, we have the following result:

Proposition 2. *Let ∇ be an ES basic fusion operator and $\Phi \mapsto \geq_\Phi$ be the basic assignment associated to ∇ by Theorem 1. Then, the following conditions hold:*

- (i) ∇ satisfies **(ESF5)** iff $\Phi \mapsto \geq_\Phi$ satisfies **1**.
- (ii) ∇ satisfies **(ESF6)** iff $\Phi \mapsto \geq_\Phi$ satisfies Properties **2**.
- (iii) ∇ satisfies **(ESF7)** iff $\Phi \mapsto \geq_\Phi$ satisfies **3**.
- (iv) ∇ satisfies **(ESF8)** iff $\Phi \mapsto \geq_\Phi$ satisfies **4**.
- (v) ∇ satisfies **(ESF8W)** iff $\Phi \mapsto \geq_\Phi$ satisfies **4'**.

The next two theorems are important representation results. They are straightforward consequences of Proposition 2.

Theorem 2 (Weak representation for ES fusion operators). *An ES combination operator ∇ is an ES fusion operator iff there exists a unique faithful assignment $\Phi \mapsto \geq_\Phi$ satisfying (B-Rep).*

Theorem 3 (Weak representation for ES quasifusion operators). *An ES combination operator ∇ is an ES quasifusion operator iff there exists a unique quasifaithful assignment $\Phi \mapsto \geq_\Phi$ satisfying (B-Rep).*

Unlike the representation theorem for IC merging operators [13,15], the previous theorems do not allow the construction of ∇ from the basic assignment. However, they allow the representation of the beliefs of the result via the total preorders of the assignment.

4 Impossibility in Epistemic Fusion

We would like the fusion process to have properties guaranteeing more global satisfaction. These good properties are based in the following principles:

- a. Any result is possible in the framework of the restrictions of the system.
- b. If all the agents have the same epistemic state it is this state which determines the result of the fusion.
- c. The fusion process depends only on how the restrictions are related in the individual epistemic states.

d. The group belief base, obtained as the result of the fusion process, does not depend on one unique agent.

Unfortunately, as we will see in this section, all these principles cannot coexist. More precisely, we will prove that for a fusion process satisfying the three first principles, necessarily there exists an agent that will impose his beliefs.

Next, we formulate the above principles in logical terms based on our operators over epistemic states.

4.1 Other social properties for fusion operators

In order to give a formulation of the principles stated above, we will suppose from now on that there are at least two propositional variables, *i.e.* the set \mathcal{P} of atomic formulas has cardinality greater or equal to two. In particular, there will be at least four interpretations in $\mathcal{W}_{\mathcal{P}}$.

The first property we present will be called *Standard Domain*. It states some “richness” in the set of results of the fusion process. This is an approximation of the first principle which says that any result is possible in the framework of the restrictions of the system.

(ESF-SD) For any i in \mathcal{S} , for any triple of interpretations w, w' and w'' in $\mathcal{W}_{\mathcal{P}}$, and any couple of epistemic states $E_{w,w'}$ and $E_{w',w''}$, such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$ and $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$, the following conditions hold:

- (i) There exists an i -profile E_i such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_{w,w'}$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w',w''}$
- (ii) There exists an i -profile E_i such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_{w,w'}$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w'}$
- (iii) There exists an i -profile E_i such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w',w''}$
- (iv) There exists an i -profile E_i such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w'}$

(ESF-SD) establishes that any result is possible when the constraints have beliefs with at most two models. Actually, this postulate is more related to the non-imposition postulate in Social Choice Theory.

Observation 1 *It is worth noting that for basic fusion operators (those satisfying Theorem 1), the satisfaction of this postulate is equivalent to the following fact: for any agent i in \mathcal{S} , any triple of interpretations w, w' and w'' in $\mathcal{W}_{\mathcal{P}}$, and any total preorder \geq between these interpretations, there is an i -profile E_i such that $\geq = \geq_{E_i} \upharpoonright_{\{w, w', w''\}}$.*

The previous Observation and some natural combinatorial arguments tell us that if the epistemic space is reduced to the consistent formulas modulo logical equivalence and the function B is the identity then the basic fusion operators satisfying the Maximality Condition can not satisfy **(ESF-SD)**. That is precisely the next result.

Theorem 4. *Consider the epistemic space $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ where $\mathcal{E} = \mathcal{L}_{\mathcal{P}}^* / \equiv$, with $\mathcal{L}_{\mathcal{P}}$ a set of formulas built over two propositional variables and $B(\varphi) = \varphi$ for every element⁶ φ in \mathcal{E} . Let $\nabla : \mathcal{P}(\mathcal{S}, \mathcal{E}) \times \mathcal{E} \rightarrow \mathcal{E}$ be an ES basic fusion operator satisfying the Maximality Condition. Then ∇ does not satisfy **(ESF-SD)**.*

The next result shows the richness and versatility of the fusion operators which satisfy this property:

Proposition 3. *Let $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ be an epistemic space, \mathcal{S} be a set of agents and ∇ an ES basic fusion operator. If ∇ satisfies **(ESF-SD)**, then for any agent i in \mathcal{S} , any i -profile E_i , any different triple of interpretations w, w', w'' in $\mathcal{W}_{\mathcal{P}}$ and any epistemic states $E_{w,w'}, E_{w,w''}, E_{w',w''}$ in \mathcal{E} , such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w,w''}) \rrbracket = \{w, w''\}$ and $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$, the following conditions hold:*

⁶ By abuse we write φ for the equivalence class of the propositional formula φ .

- (i) For all j in \mathcal{S} , there exists a j -profile E_j such that
 - $B(\nabla(E_j, E_{w,w'})) \equiv B(\nabla(E_j, E_{w,w''})) \equiv \varphi_w$, and
 - $B(\nabla(E_j, E_{w',w''})) \equiv B(\nabla(E_i, E_{w',w''}))$.
- (ii) For all k in \mathcal{S} , there exists a k -profile E_k such that
 - $B(\nabla(E_k, E_{w,w'})) \equiv \varphi_w$,
 - $B(\nabla(E_k, E_{w',w''})) \equiv \varphi_{w'}$, and
 - $B(\nabla(E_k, E_{w,w''})) \equiv B(\nabla(E_i, E_{w,w''}))$.

The following two properties try to capture the meaning of the second of the principles above mentioned. The first of them is given in terms of similarity between the epistemic states of the agents. Actually, it is natural to think that if the agents totally coincide *i.e.* all have the same epistemic state, then the result of the group can be determined for any individual agent. More precisely, we have the following postulate called *Unanimity condition*:

(ESF-U) For all N in $\mathcal{F}^*(\mathcal{S})$, for each N -profile Φ and for every epistemic state E in \mathcal{E} , if $E_i = E_j$, for any pair i, j in N , then, for all i in N , $B(\nabla(\Phi, E)) \equiv B(\nabla(E_i, E))$.

(ESF-U) establishes that if all the agents in the society have the same epistemic state, then the result of the fusion is exactly the fusion (or the revision⁷) of the society conformed by any of the agents given the integrity constraints.

The following result is important. It establishes that the ES fusion operators satisfy the Unanimity condition.

Proposition 4. Let ∇ be an ES combination operator. If ∇ satisfies **(ESF2)**, **(ESF7)** and **(ESF8)**, it satisfies **(ESF-U)**.

The converse of this result does not hold, as can be seen in Section 5.

In the following result we give a semantic characterization of the Unanimity condition.

Proposition 5. Let ∇ be an ES basic fusion operator. Then the following statements are equivalent:

- (i) ∇ satisfies **(ESF-U)**.
- (ii) For all N in $\mathcal{F}^*(\mathcal{S})$, for each N -profile Φ , for every epistemic state E in \mathcal{E} such that $|\llbracket B(E) \rrbracket| \leq 2$, if $E_i = E_j$ for each pair i, j in N , then $B(\nabla(\Phi, E)) \equiv B(\nabla(E_i, E))$, for all i in N .
- (iii) The assignment $\Phi \mapsto \geq_\Phi$, representing ∇ , satisfies the following property:
 - (u) For each society N , for each N -profile Φ , if $E_i = E_j$ for each couple i, j in N , then $\geq_\Phi = \geq_{E_i}$ for all i in N .

Note that this last condition is closer to the formulations of Unanimity in Social Choice Theory.

Now we give a second syntactical formulation of the second principle: if all the agents reject a given alternative, this alternative will be rejected in the result of fusion. This form of excluding unanimity is known in Social Choice Theory like *Pareto condition*. We keep this name for the following postulate:

(ESF-P) For all N in $\mathcal{F}^*(\mathcal{S})$, for each N -profile Φ , for all epistemic states E and E' in \mathcal{E} , if $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$ and $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in N , then $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.

⁷ When the society is formed by one unique agent, it is well known that the process of fusion corresponds to a revision of the epistemic state of the agent by the integrity constraints (see for instant [15]).

The following result entails that ES fusion operators satisfy the Pareto condition:

Proposition 6. *If ∇ is an ES combination operator that satisfies (ESF7) and (ESF8W), then ∇ also satisfies (ESF-P).*

The converse is not true as will be seen in Section 5.

The Pareto condition has also a semantical characterization that we give in the next result:

Proposition 7. *Let ∇ be an ES basic fusion operator. Then the following statements are equivalent:*

- (i) ∇ satisfies (ESF-P)
- (ii) For all N in $\mathcal{F}^*(\mathcal{S})$, for every N -profile Φ , for all epistemic states E and E' in \mathcal{E} such that $\|B(E)\| \leq 2$, if $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in N , then $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$
- (iii) The assignment $\Phi \mapsto \succeq_\Phi$, representing ∇ , satisfies the following property:
 - (p) For all N in $\mathcal{F}^*(\mathcal{S})$, for each N -profile Φ , for all interpretations w y w' , if $w \succ_{E_i} w'$ for all i in N , then $w \succ_\Phi w'$

We continue stating the syntactical postulate aiming to catch the third principle: *the fusion process depends only on how the restrictions in the individual epistemic states are related.*

In Social Choice Theory this idea is captured by the postulate known as *Independence of Irrelevant Alternatives*. In our framework this postulate will be stated in the following manner:

(ESF-I) *For all N in $\mathcal{F}^*(\mathcal{S})$, for all N -profiles Φ and Φ' , for every epistemic state E we have $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$, whenever for each epistemic state E' such that $B(E') \vdash B(E)$, we have $B(\nabla(E_j, E')) \equiv B(\nabla(E'_j, E'))$ for all $j \in N$.*

This property, called *Independence condition*, essentially says the following: Given an integrity constraint, if each agent in the fusion process has two possible choices of epistemic states, and if revising these epistemic states by integrity constraints having beliefs stronger than the given integrity constraint, the beliefs of the epistemic states resulting coincide, then the result of the fusion of the society of agents under the given integrity constraint is the same, at the level of beliefs, for any choice of the epistemic state made by each agent.

The next result gives a simplification of the Postulate (ESF-I) for the ES basic fusion operators and also a semantic characterization.

Proposition 8. *Let ∇ be an ES basic fusion operator. Then the following statements are equivalent:*

- (i) ∇ satisfies (ESF-I)
- (ii) For all N in $\mathcal{F}^*(\mathcal{S})$, for all N -profiles Φ and Φ' , for all E in \mathcal{E} such that $\|B(E)\| \leq 2$, if $B(\nabla(E_j, E)) \equiv B(\nabla(E'_j, E))$, for all j in N , then $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$
- (iii) The assignment $\Phi \mapsto \succeq_\Phi$, representing ∇ , satisfies the following property:
 - (ind) For all N in $\mathcal{F}^*(\mathcal{S})$, for all N -profiles Φ and Φ' , for all interpretations w and w' , if $\succeq_{E_i} \upharpoonright_{\{w, w'\}} = \succeq_{E'_i} \upharpoonright_{\{w, w'\}}$, for all i in N , then $\succeq_\Phi \upharpoonright_{\{w, w'\}} = \succeq_{\Phi'} \upharpoonright_{\{w, w'\}}$

It is worth noting that in presence of (ESF-I) and the basic postulates, Postulate (ESF-U) entails (ESF-P), i.e. unanimity is stronger than Pareto condition. That is the following proposition:

Proposition 9. *If an ES basic fusion operator ∇ satisfies (ESF-U) and (ESF-I), then it also satisfies (ESF-P).*

The converse is not true, that is, even in presence of **(ESF-I)**, Postulate **(ESF-P)** does not entail **(ESF-U)**. Actually, it is not difficult to build operators satisfying **(ESF-P)**, **(ESF-I)** but for which **(ESF-U)** does not hold (see Section 5). Moreover, as we can see in Section 5, **(ESF-P)** does not entail **(ESF-I)**, even if **(ESF-U)** holds.

Next we state the postulate related with the fourth principle: *the group belief base, obtained as the result of the fusion process, does not depend on one unique agent*. Actually, we establish the negative form which says that there is an agent that imposes his will, a *dictatorial agent*. This is the postulate that the *good* operators should avoid.

(ESF-D) For all N in $\mathcal{F}^*(S)$ there exists an agent d_N in N such that, for all N -profile Φ and for all epistemic state E in \mathcal{E} , $B(\nabla(\Phi, E)) \vdash B(\nabla(E_{d_N}, E))$.

The operators satisfying the previous postulate are called *Dictatorial operators*. Suppose that the operator ∇ is dictatorial, then given N in $\mathcal{F}^*(S)$, an agent d_N satisfying the property in **(ESF-D)**, is called a *dictator* in N or simply *N-dictator*, with respect to ∇ .

The following result is a semantic characterization of Dictatorial operators:

Proposition 10. Let ∇ be an ES basic fusion operator. Then the following statements are equivalent:

- (i) ∇ is dictatorial.
- (ii) For all N in $\mathcal{F}^*(S)$, there exists d_N in N such that, for every N -profile Φ and for all epistemic state E in \mathcal{E} , if $|\llbracket B(E) \rrbracket| \leq 2$ then $B(\nabla(\Phi, E)) \vdash B(\nabla(E_{d_N}, E))$.
- (iii) The assignment $\Phi \mapsto \geq_\Phi$, representing ∇ , satisfies the following property:
 - (d) For all N in $\mathcal{F}^*(S)$, there exists d_N in N such that for all interpretations w and w' , if $w >_{E_{d_N}} w'$, then $w >_\Phi w'$.

The next result establishes that Dictatorial operators satisfy the Pareto Condition.

Proposition 11. Let ∇ be an ES combination operator. If ∇ satisfies **(ESF-D)**, then ∇ satisfies **(ESF-P)**.

However, there are dictatorial operators for which **(ESF-U)** does not hold (see Section 5).

4.2 Coalitions and decisional power

In many situations a group of agents or individuals who integrate a society form alliances or coalitions in order to achieve common goals. Thus, in a first step to define important types of alliances, we will say simply that a *coalition* is determined by a group of individuals of a society. More precisely, we state:

Definition 10 (Coalitions). Given a finite society of agents N in $\mathcal{F}^*(S)$, a *coalition of agents in N* , or simply an *N-coalition*, is a set D of agents in N .

Some coalitions in logic-based fusion have an important decisional power. This kind of decisional power can be illustrated as follows: Let us suppose that the agents of a society have to decide about the acceptance of certain information under some given condition (integrity constraints). Also suppose that, under such restriction, the new information does not satisfy some of the agents (the coalition) at all, and the rest of the agents accept this information. When the global decision is to reject the new information, it is clear that the coalition has imposed its view. This kind of coalitions, with great decisional power, are called *locally decisive coalitions*.

Definition 11 (Locally decisive coalitions). Let N be in $\mathcal{F}^*(S)$, ∇ be an ES combination operator, E, E' be a pair of epistemic states and D be an N -coalition. We say that D is a locally decisive N -coalition for E against E' , with respect to ∇ , denoted by $ED^\nabla E'$, if for each N -profile Φ satisfying:

- (i) $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in D ;
- (ii) $B(\nabla(E_j, E)) \equiv B(E')$, for all j in $N \setminus D$; and
- (iii) $\bigwedge_{i \in D} B(\nabla(E_i, E)) \not\vdash \perp$

we have that $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.

It is also possible that the rejection of the new information will be achieved by a coalition independently from the opinion of the rest of the agents involved in the merging process. This kind of coalitions are called *decisive coalitions*.

Definition 12 (Decisive coalition). Let N be in $\mathcal{F}^*(S)$, ∇ be an ES combination operator, E, E' be a pair of epistemic states and D be an N -coalition. We say that D is a decisive N -coalition for E against E' , with respect to ∇ , denoted by $ED^{\nabla^*} E'$, if for every N -profile Φ satisfying:

- (i) $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in D ; and
- (ii) $\bigwedge_{i \in D} B(\nabla(E_i, E)) \not\vdash \perp$

we have that $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.

We will say that D is a decisive N -coalition, with respect to ∇ , if $ED^{\nabla^*} E'$ for any pair E, E' of epistemic states.

Observation 2 For any finite society N in $\mathcal{F}^*(S)$, any ES combination operator ∇ , any N -coalition D and any tuple of epistemic states E, E', \bar{E}, \bar{E}' in \mathcal{E} , it is easy to see that the following conditions hold:

- a. If $ED^{\nabla^*} E'$, then $ED^\nabla E'$
- b. If ∇ satisfies (ESF1) and $B(E) \equiv B(E')$, or $B(E)$ is inconsistent with $B(E')$, then $ED^{\nabla^*} E'$ and therefore $ED^\nabla E'$
- c. If ∇ satisfies (ESF2), $B(E) \equiv B(\bar{E})$ and $B(E') \equiv B(\bar{E}')$, then $ED^{\nabla^*} E'$ if, and only if, $\bar{E}D^{\nabla^*} \bar{E}'$
- d. If ∇ satisfies (ESF1) and D is decisive, then D is nonempty⁸.
- e. If ∇ satisfies (ESF-P), then N is a decisive N -coalition by itself.

The following result is important:

Proposition 12. Let ∇ be a ES combination operator. Let N be a finite society of agents. If D is a decisive N -coalition and its cardinality is one, then the only element of D is a dictator in N .

Next, we introduce the first result about decisive coalition. It establishes some necessary and sufficient conditions that must be satisfied by a basic assignment, associated to certain ES basic fusion operator, in order for it to admit a decisive coalition. Due to the similarity between the Pareto condition and the decisive coalition notion, the proof of this result is analogue to that of Proposition 7.

Proposition 13. Given N in $\mathcal{F}^*(S)$, ∇ an ES basic fusion operator and $\Phi \mapsto \geq_\Phi$ the basic assignment associated to ∇ , the following claims about an N -coalition D are equivalent:

⁸ If $D = \emptyset$ the two premisses in Definition 12 are always true. Thus, if D were decisive, taking $E = E'$, we get $B(\nabla(\Phi, E)) \wedge B(E) \vdash \perp$, a contradiction.

- (i) D is a decisive N -coalition, with respect to ∇ .
- (ii) For each N -profile Φ and any pair of epistemic states E, E' in \mathcal{E} , if $|\llbracket B(E) \rrbracket| \leq 2$ and $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in D , then $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.
- (iii) For each N -profile Φ , and all interpretations w, w' , if $w \succ_{E_i} w'$, for all i in D , then $w \succ_{\Phi} w'$.

Let us note that the last result, in addition to giving a semantic characterization of operators that admit decisive coalitions (point (iii)), establishes that in order to determine the decisive nature of a coalition with respect to an ES basic operator, it is enough considering epistemic states whose most entrenched beliefs have at most two models (point (ii)).

Under certain rational properties over a merging process, if a coalition is locally decisive between a pair of epistemic states of a certain kind, then we can change slightly the pair of epistemic states and the coalition will continue to be locally decisive for this new pair of epistemic states. More precisely, we have the following two results:

Proposition 14 (First propagation lemma). *Given N in $\mathcal{F}^*(S)$, an N -coalition D , an ES basic fusion operator ∇ satisfying (ESF-SD), (ESF-P) and (ESF-I), and w, w' a pair of different interpretations. If $E_{w,w'}, E_{w'}$ are epistemic states such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w'}) \rrbracket = \{w'\}$ and $E_{w,w'} D^{\nabla} E_{w'}$, then, for any interpretation w'' , with $w'' \not\models B(E_{w,w'})$ and any pair of epistemic states $E_{w,w''}, E_{w''}$, with $\llbracket B(E_{w,w''}) \rrbracket = \{w, w''\}$ and $\llbracket B(E_{w''}) \rrbracket = \{w''\}$, we have $E_{w,w''} D^{\nabla} E_{w''}$.*

Proposition 15 (Second propagation lemma). *Given N in $\mathcal{F}^*(S)$, an N -coalition D , an ES basic fusion operator ∇ satisfying (ESF-SD), (ESF-P) and (ESF-I), and w, w' a pair of different interpretations. If $E_{w,w'}, E_{w'}$ are epistemic states such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w'}) \rrbracket = \{w'\}$ and $E_{w,w'} D^{\nabla} E_{w'}$, then, for any interpretation w'' such that $w'' \not\models B(E_{w,w'})$ and any pair of epistemic states $E_{w',w''}$ and $E_{w''}$ such that $\llbracket B(E_{w',w''}) \rrbracket = \{w, w''\}$ and $\llbracket B(E_{w''}) \rrbracket = \{w''\}$, we have $E_{w',w''} D^{\nabla} E_{w''}$.*

The previous two propagation results will allow us to show that if a coalition is locally decisive with respect to a given pair of epistemic states then such coalition is decisive. More precisely we have the following theorem:

Theorem 5 (Generalized propagation theorem). *Consider N in $\mathcal{F}^*(S)$, let ∇ be an ES basic fusion operator satisfying (ESF-SD), (ESF-P) and (ESF-I), and D be an N -coalition. If there exist different interpretations w and w' , and epistemic states $E_{w,w'}$ and $E_{w'}$, such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w'}) \rrbracket = \{w'\}$ and $E_{w,w'} D^{\nabla} E_{w'}$, then D is a decisive coalition.*

4.3 Impossibility results

There exist some links between Weak Standard Domain, Unanimity, Pareto, Independence conditions and the fact that a merging process admits a dictatorial agent. Such links are revealed through the Main Impossibility Theorem below. This theorem shows that if Weak Standard Domain, Pareto and Independence conditions hold, an ES basic fusion operator is really a dictatorial operator.

Theorem 6 (Main impossibility theorem). *Let ∇ be an ES basic fusion operator. If ∇ satisfies (ESF-SD), (ESF-P) and (ESF-I), then (ESF-D) holds.*

Note that the condition of Transitive explanations in Social Choice Theory is actually implicit in the representation of ES basic fusion operators.

Postulates **(ESF-SD)**, **(ESF-P)** and **(ESF-I)** are only sufficient conditions in order to have an ES basic fusion operator be dictatorial. Actually, as we will see in Section 5, there exist ES basic fusion operators that satisfy

(ESF-D) but do not satisfy **(ESF-I)**, namely the Σ -pseudoprojective ES fusion operators. There are also dictatorial operators which do not satisfy **(ESF-SD)**, namely the linearized projective operators (see Section 5).

By the connection between the postulates established in Section 3 and the Pareto condition which has been revealed in Proposition 6 and the Main Impossibility Theorem we have:

Corollary 1. *Any ES basic fusion operator that satisfies **(ESF-SD)**, **(ESF7)**, **(ESF8W)** (or **(ESF8)**) and **(ESF-I)**, also satisfies **(ESF-D)**.*

Similarly, by virtue of the relationship between the Unanimity and Pareto conditions, from Proposition 9 and the Main Impossibility Theorem we have the following corollary:

Corollary 2. *Any ES basic fusion operator that satisfies **(ESF-SD)**, **(ESF-U)** and **(ESF-I)**, also satisfies **(ESF-D)**.*

5 Concrete examples of ES basic fusion operators

In this section we assume that the epistemic states are total preorders over interpretations and we have $\llbracket B(\geq) \rrbracket = \max(\geq)$ for any epistemic state \geq . Then, with this concrete representation of epistemic states, we will define some ES basic fusion operators.

5.1 Some classes of ES basic fusion operators

First, we are going to define a class of ES basic fusion operators using aggregation functions. These operators are called *aggregation-based ES fusion operators*. In order to do this, let us recall the definition of aggregation functions.

Definition 13 (Aggregation functions). *An (symmetric) aggregation function F is a total function associating a nonnegative real number to every finite tuple of nonnegative integers such that for any x_1, \dots, x_n, x, y in \mathbb{Z}^+ :*

Monotony $F(x_1, \dots, x, \dots, x_n) \geq F(x_1, \dots, y, \dots, x_n)$, if $x \geq y$

Minimality $F(x_1, x_2, \dots, x_n) = 0$ if and only if $x_1 = x_2 = \dots = x_n = 0$

Identity $F(x) = x$

Symmetry For any permutation σ , $F(x_1, x_2, \dots, x_n) = F(\sigma(x_1, x_2, \dots, x_n))$

Two examples of aggregation functions are *sum* and *max*. They are classic examples in the study of logic-based fusion [13].

sum $\sum(x_1, x_2, \dots, x_n) = \sum x_i$

max $\max(x_1, x_2, \dots, x_n) = \max\{x_1, x_2, \dots, x_n\}$

An aggregation function F induces a total preorder over \mathcal{W}_Φ , for any profile of epistemic states as follows:

For any $N = \{i_1, i_2, \dots, i_n\}$ in $\mathcal{F}^*(\mathcal{S})$, any N -profile $\Phi = (\succeq_{i_1}, \succeq_{i_2}, \dots, \succeq_{i_n})$, and any couple of interpretations w, w' in \mathcal{W}_Φ

$$w \succeq_\Phi^F w' \text{ iff } F(r_{i_1}(w), \dots, r_{i_n}(w)) \geq F(r_{i_1}(w'), \dots, r_{i_n}(w')) \quad (3)$$

where $r_i(x)$ is the rank of an interpretation x in the total preorder \succeq_i , for each agent i in N .

It is easy to see that, for a given profile of epistemic states Φ , the relation \succeq_Φ^F is a total preorder over \mathcal{W}_Φ . Thus, for any aggregation function F it is possible to define an assignment $\Phi \mapsto \succeq_\Phi^F$ which *preserves the structure of epistemic states*, that is, for all i in \mathcal{S} , $\succeq_{\succeq_i} = \succeq_i$ for any total i -profile (epistemic state) \succeq_i . When an assignment preserves the structure of epistemic states we also say that the assignment is *structure preserving*. Note that these assignments trivially are basic assignments that satisfy Property 1.

From the definition (3), in addition to the equations (1) and (B-Rep), it is possible to build an ES basic fusion operator ∇^F associated to an aggregation function F as follows:

$$\nabla^F(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi^F)} \quad (4)$$

Thanks to the equation above we define two aggregation-based ES basic fusion operators: ∇^Σ (*sum*) and ∇^{\max} (*max*), defined from the aggregation functions *sum* and *max* respectively.

Another class of ES combination operators are given in terms of projections or variants of this. The operators in this class are called *projective-based ES fusion operators*. In order to define this class, from now on, d_N denotes $\max(N)$ for each finite society N of agents in \mathcal{S} .

The first one of such operators is determined by projections over profiles of epistemic states. This operator is called *projective ES basic fusion operator*, ∇^π (*projective operator* by abuse of notation), and is defined as follows:

Projective ES basic fusion operator

$$\nabla^\pi(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_{d_N})}$$

For this operator, the output of a merging process depends completely on the beliefs of a single agent involved in such merging process and the integrity constraints.

Two variants of the projective operator are obtained through a “linearization” of it. These families of operators are called *linearized projective ES basic fusion operators* and *quasilinearized projective ES basic fusion operators* (called respectively *linearized projective operators* and *quasilinearized projective operators* by abuse). In order to define these operators, let us consider \succeq , a fixed linear order over \mathcal{W}_Φ :

Linearized projective ES basic fusion operator

$$\nabla^{\pi_\succeq}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi^{\pi_\succeq})}$$

where $\succeq_\Phi^{\pi_\succeq} = \succeq^{\text{lex}(\succeq_{d_N}, \succeq)}$.

Quasilinearized projective ES basic fusion operator

$$\nabla^{Q\pi_\succeq}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi^{Q\pi_\succeq})}$$

where $\succeq_{\Phi}^{Q\pi_z} = \succeq_i$ if Φ has a single input, namely \succeq_i , and $\succeq_{\Phi}^{Q\pi_z} = \succeq^{\text{lex}(\succeq_{d_N}, \succeq)}$ otherwise.

The following projective-based ES fusion operator is defined using sum and projections. This operator will be called Σ -pseudoprojective ES fusion operator, $\nabla^{\Sigma-P\pi}$:

Σ -pseudoprojective ES fusion operator

$$\nabla^{\Sigma-P\pi}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_{\Phi}^{\Sigma-P\pi})}$$

where $\succeq_{\Phi}^{\Sigma-P\pi} = \succeq^{\text{lex}(\succeq_{d_N}, \succeq_{\Phi}^{\Sigma})}$.

In this kind of merging process an agent allows a very weak participation of all agents involved in the merging process.

Like sum and max operators, the projective, the linearized projective, the quasilinearized projective and Σ -pseudoprojective operators are ES basic fusion operators, being $\Phi \mapsto \succeq_{d_N}$, $\Phi \mapsto \succeq_{\Phi}^{\pi_z}$, $\Phi \mapsto \succeq_{\Phi}^{Q\pi_z}$ and $\Phi \mapsto \succeq_{\Phi}^{\Sigma-P\pi}$ their respective basic assignments.

5.2 Behavior of our examples

In this section we make an exhaustive study of the rational behavior of the previously defined operators (sum, max, projective, linearized projective, quasilinearized projective and Σ -pseudoprojective operators). In particular, we show which postulates from the set of postulates previously established are satisfied.

From the properties of sum and max it is easy to see that sum operator is really an ES fusion operator, unlike max operator which is simply an ES quasifusion operator because it satisfies (ESF8W), but (ESF8) does not hold.

Also let us note that operators defined by aggregation functions satisfy (ESF-SD). This is because they are structure preserving and the freedom one has for building different preorders through its assignment.

Note that the sum operator satisfies (ESF-U), since (ESF7) and (ESF8) hold (see Proposition 4). Moreover, it is easy to verify that the max operator also satisfies the unanimity condition but does not satisfy (ESF8), showing that (ESF7) and (ESF8) are together a sufficient but not a necessary condition for (ESF-U) to hold.

Also, sum and max operators satisfy (ESF-P). This is because (ESF7) and (ESF8W) hold (see Proposition 6).

Furthermore, under the symmetry property of aggregation functions, it is easy to see that aggregation-based ES basic fusion operators are not dictatorial operators. In particular, sum and max operators are not dictatorial operators. Thus, by the Main Impossibility Theorem, these operators do not satisfy (ESF-I), because (ESF-SD) and (ESF-P) hold. Moreover, since these operators satisfy (ESF-U) and (ESF-P) but do not satisfy (ESF-I), we have that, modulo the Unanimity condition, the Pareto condition does not entail Independence condition; that is, one of the converses of Proposition 9 does not hold.

The projective operator satisfies (ESF5), (ESF7) and (ESF8W), but (ESF6) and (ESF8) do not hold, as we show in Proposition 16 (see A). Otherwise, as we can see in Proposition 17 (see A), linearized projective operators satisfy (ESF7), (ESF8) and (ESF8W), but do not satisfy neither (ESF5) nor (ESF6). Moreover, as we show in Proposition 18 (see A), quasilinearized projective operators satisfy (ESF5) and (ESF8W) but do not satisfy neither (ESF6), (ESF7) nor (ESF8). The Σ -pseudoprojective operator satisfies (ESF5) and (ESF6), but (ESF7), (ESF8) and (ESF8W) do not hold (see A, Proposition 19).

In Propositions 16, 17, 18 and 19, we also show that the projective-based ES fusion operators previously defined are dictatorial operators, that is, satisfy (ESF-D). Thus, by Proposition 11, the operators in this

Table 1. Properties of the ES basic fusion operators previously defined

Operator	(ESF5)	(ESF6)	(ESF7)	(ESF8)	(ESF8W)	(ESF-SD)	(ESF-U)	(ESF-P)	(ESF-I)	(ESF-D)
∇^Σ	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗
∇^{\max}	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗
∇^π	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓
$\nabla^{\pi \geq}$	✗	✗	✓	✓	✓	✗	✓	✓	✓	✓
$\nabla^{Q\pi \geq}$	✓	✗	✗	✗	✓	✓	✗	✓	✓	✓
$\nabla^{\Sigma-P\pi}$	✓	✓	✗	✗	✗	✓	✓	✓	✗	✓

class of ES basic fusion operators also satisfy the Pareto condition, that is, **(ESF-P)** holds. Moreover, in these results we can see that the projective, quasilinearized projective and the Σ -pseudoprojective operators satisfy **(ESF-SD)**, unlike linearized projective operators that do not satisfy it. There we also show that the projective, linearized and quasilinearized projective operators satisfy the Independence condition, **(ESF-I)**, unlike the Σ -pseudoprojective operator which does not satisfy it.

Furthermore, Propositions 16, 17 and 19 also show respectively that, the projective, linearized projective and the Σ -pseudoprojective operators satisfy the Unanimity condition, **(ESF-U)**, unlike quasilinearized projective operators which do not satisfy it, as we show in Proposition 18. Thus, satisfaction of the Pareto condition does not imply satisfaction of the Unanimity condition, even if the Independence condition holds, that is, the other converse of Proposition 9 does not hold.

Table 1 shows the properties satisfied by the previously defined operators in a synthesized way, offering a landscape of the rational properties that satisfy those operators. In this table, and for the operators pointed in the first column, ✓ shows the satisfaction of the property indicated at the top of its column, while ✗ denotes that such property does not hold.

As the Main impossibility theorem establishes, in order to see that an ES fusion basic operator is dictatorial, it is enough to see that it satisfies **(ESF-SD)**, **(ESF-P)** and **(ESF-I)**. However, as shown in Table 1, the Σ -pseudoprojective operator is an example of a dictatorial operator that does not satisfy the Independence condition, **(ESF-I)**. Furthermore, we can see that linearized projective operators are dictatorial operators which do not satisfy **(ESF-SD)**. All this shows in a finer way that these three properties are not necessary conditions for an ES basic fusion operator to be dictatorial.

6 Final remarks and perspectives

We have presented an epistemic version of postulates of merging which allow us to give very precise representation theorems (Theorems 1, 2, 3 and Proposition 1). We have also introduced new postulates with a more social flavor. These postulates, except Standard Domain, have been characterized semantically (Propositions 5, 7, 8, 10). We have shown some tight relationships between the merging postulates and the social postulates; in particular, the ES quasifusion operators satisfy the Pareto Condition (Proposition 6) and the ES fusion operators satisfy the Unanimity Condition (Proposition 4). Actually, unlike the other social postulates the Standard Domain Postulate has a formulation which is very near to a semantical one as the Observation 1 establishes.

The social postulates together with the basic fusion postulates are enough to prove a general impossibility theorem: the ES basic fusion operator satisfying the Standard Domain, Pareto and Independence Conditions are indeed dictatorial operators (Theorem 6). Moreover, the ES fusion operators satisfying the Standard Domain Postulate and the Independence Condition are dictatorial operators (Corollary 1).

One very interesting feature of our approach is that it gives interesting instantiations of the Impossibility Theorem (Theorem 6) for different representations of epistemic states. Thus, this applies to Ordinal Conditional Functions, rational relations, and of course total preorders. However, the representation of epistemic states as formulas does not work because it is impossible to have Standard Domain in presence of a good representation of beliefs, namely the Maximality Condition. This fact highlights the necessity of using complex epistemic states.

The concrete examples of Section 5 show that the Standard Domain and Independence are not necessary conditions in order to have a dictatorial operator (see Table 1).

Having established the impossibility Theorem, it is natural to ask for general results of manipulability. We have some work in progress in this direction.

At the moment, we have not an example of an ES fusion or quasifusion operator which is at the same time a dictatorial operator. We conjecture that such kind of operators exist.

We think that Theorem 4 can be generalized to an arbitrary number of models greater than 4. However the combinatorial analysis seems very complicated.

Finally, an interesting question is to find a characterization of dictatorial operators.

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A Proofs

Proof of Proposition 1: Let $\Phi \mapsto \geq_\Phi$ be an assignment that satisfies Properties **3** and **4**. If this assignment satisfies **2** the maximality condition follows straightforwardly. In order to prove the converse, we suppose that $\Phi \mapsto \geq_\Phi$ satisfies the maximality condition, and using induction on the length of finite societies, we will show that for any N in $\mathcal{F}^*(S)$, N satisfies the following:

$$\text{For any } N\text{-profile } \Phi, \text{ if } \bigwedge_{i \in N} B(E_i) \not\models \perp, \text{ then } \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket = \max(\geq_\Phi) \quad (5)$$

If N has a single agent, the above property is exactly the maximality condition. Now, let n be a positive integer, with $n > 1$, and suppose that for all D in $\mathcal{F}^*(S)$, with less than n agents, the condition (5) holds. Assume N in $\mathcal{F}^*(S)$ consisting of n agents. Let us consider Φ an N -profile satisfying $\bigwedge_{i \in N} B(E_i) \not\models \perp$. Let w and w' be interpretations. Assume that $w \models \bigwedge_{i \in N} B(E_i)$. Let us take $D = N \setminus \{j\}$ a finite society of agents in S , where j is an agent in N . Thus, the D -profile $\Phi \upharpoonright_D$ satisfies $w \models \bigwedge_{i \in D} B(E_i)$. Then, by condition (5), $w \geq_{\Phi \upharpoonright_D} w'$. Since $w \models B(E_j)$, by the maximality condition, we have $w \geq_{E_j} w'$. Hence, from Property **3**, it follows that $w \geq_\Phi w'$, that is, $w \in \max(\geq_\Phi)$. Thus, $\llbracket \bigwedge_{i \in N} B(E_i) \rrbracket \subseteq \max(\geq_\Phi)$.

Now we prove that $\max(\geq_\Phi) \subseteq \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket$. Let w be in $\max(\geq_\Phi)$ and, towards a contradiction, suppose $w \not\models \bigwedge_{i \in N} B(E_i)$. Thus, there exists j in N such that $w \not\models B(E_j)$. Let us consider the finite society $D = N \setminus \{j\}$. Take an interpretation w' such that $w' \models \bigwedge_{i \in N} B(E_i)$. Clearly, we have $w' \models \bigwedge_{i \in D} B(E_i)$. From condition (5), we obtain $w' \geq_{\Phi \upharpoonright_D} w$. Since $w' \models B(E_j)$ and $w \not\models B(E_j)$, from the maximality condition we have $w' >_{E_j} w$. Therefore, by Property **4**, we have $w' >_\Phi w$, a contradiction. \square

Proof of Theorem 1:

(Only if part) Assume that ∇ is an ES basic fusion operator. We define the assignment $\Phi \mapsto \geq_\Phi$ associated to this operator as follow: for all N in $\mathcal{F}^*(S)$ and every N -profile Φ , \geq_Φ is given by:

$$w \geq_\Phi w' \text{ iff } w \models B(\nabla(\Phi, E_{w, w'})); \quad (6)$$

where $E_{w, w'}$ is an epistemic state in \mathcal{E} satisfying $\llbracket B(E_{w, w'}) \rrbracket = \{w, w'\}$.

Since $B(\mathcal{E})$ is exactly $\mathcal{L}_{\mathcal{P}}^*/\equiv$, modulo equivalence, there exists such an epistemic state $E_{w,w'}$ in \mathcal{E} . Moreover, because of **(ESF2)**, the definition of \geq_{Φ} does not depend on the choice of $E_{w,w'}$. Thus, let us show that \geq_{Φ} is a total preorder over $\mathcal{W}_{\mathcal{P}}$.

Total Let w, w' be interpretations in $\mathcal{W}_{\mathcal{P}}$ and consider $E_{w,w'}$ an epistemic state that satisfies $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$. From **(ESF1)** and the consistency of $B(\nabla(\Phi, E_{w,w'}))$, we have either $w \models B(\nabla(\Phi, E_{w,w'}))$ or $w' \models B(\nabla(\Phi, E_{w,w'}))$. By definition, this means that $w \geq_{\Phi} w'$ or $w' \geq_{\Phi} w$, respectively.

Transitivity Assume that w, w' and w'' are interpretations in $\mathcal{W}_{\mathcal{P}}$ and suppose that $w \geq_{\Phi} w'$ and $w' \geq_{\Phi} w''$. We want to show that $w \geq_{\Phi} w''$. Towards a contradiction, assume that $w \not\geq_{\Phi} w''$. Let $E_{w,w'}, E_{w',w''}, E_{w,w''}$, E be epistemic states satisfying $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$, $\llbracket B(E_{w,w''}) \rrbracket = \{w, w''\}$ and $\llbracket B(E) \rrbracket = \{w, w', w''\}$. Since $B(\nabla(\Phi, E_{w,w'}))$ is consistent and $w \not\geq_{\Phi} w''$, from **(ESF1)** we have that w'' is the unique model of $B(\nabla(\Phi, E_{w,w'}))$. Now we consider the following two cases:

- $B(\nabla(\Phi, E)) \wedge B(E_{w,w''}) \not\models \perp$. In this case, since $B(E_{w,w''}) \equiv B(E) \wedge B(E_{w,w'})$, by **(ESF3)** and **(ESF4)**, we have

$$B(\nabla(\Phi, E)) \wedge B(E_{w,w''}) \equiv B(\nabla(\Phi, E_{w,w''})) \quad (7)$$

From the fact that $w \not\models B(\nabla(\Phi, E_{w,w''}))$ and the equivalence (7), we have $w \not\models B(\nabla(\Phi, E))$. From the equivalence (7), we have also $w'' \models B(\nabla(\Phi, E))$. Then, by **(ESF3)** and **(ESF4)** we have

$$B(\nabla(\Phi, E)) \wedge B(E_{w',w''}) \equiv B(\nabla(\Phi, E_{w',w''})) \quad (8)$$

Moreover, $w' \geq_{\Phi} w''$ means that $w' \models B(\nabla(\Phi, E_{w',w''}))$. So, from the equivalence (8), we have that $w' \models B(\nabla(\Phi, E))$. Therefore, $\llbracket B(\nabla(\Phi, E)) \rrbracket = \{w', w''\}$. Thus, by **(ESF3)** and **(ESF4)**, we have

$$B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \equiv B(\nabla(\Phi, E_{w,w'})) \quad (9)$$

But $w \geq_{\Phi} w'$, then, by definition and the equivalence (9), we have $w \models B(\nabla(\Phi, E))$, a contradiction.

- $B(\nabla(\Phi, E)) \wedge B(E_{w,w''}) \models \perp$. In this case, by **(ESF1)**, w' is the sole model of $B(\nabla(\Phi, E))$. Therefore, $B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \not\models \perp$ and by **(ESF3)** and **(ESF4)**, we have

$$B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \equiv B(\nabla(\Phi, E_{w,w'})) \quad (10)$$

Since $w \not\models B(\nabla(\Phi, E))$, by the equivalence (10), we have $w \not\models B(\nabla(\Phi, E_{w,w'}))$, that is, $w \not\geq_{\Phi} w'$, a contradiction.

Now we show that $\Phi \mapsto \geq_{\Phi}$ satisfies (B-Rep). In order to do this, we consider N in $\mathcal{F}^*(S)$, an N -profile Φ and an epistemic state E in \mathcal{E} and we have to verify that $\llbracket B(\nabla(\Phi, E)) \rrbracket = \max(\llbracket B(E) \rrbracket, \geq_{\Phi})$.

- First we prove that $\llbracket B(\nabla(\Phi, E)) \rrbracket \subseteq \max(\llbracket B(E) \rrbracket, \geq_{\Phi})$. Let w be an interpretation in $\mathcal{W}_{\mathcal{P}}$ such that $w \models B(\nabla(\Phi, E))$. By **(ESF1)**, $w \models B(E)$. Towards a contradiction, suppose that w is not in $\max(\llbracket B(E) \rrbracket, \geq_{\Phi})$. Thus, there exists w' , a model of $B(E)$ such that $w' >_{\Phi} w$. Let $E_{w,w'}$ be an epistemic state which satisfies $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$. By the definition given by the equivalence (6), we have $w \not\models B(\nabla(\Phi, E_{w,w'}))$. Then, since $B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \not\models \perp$ and $B(E_{w,w'}) \equiv B(E) \wedge B(E_{w,w'})$, by **(ESF3)** and **(ESF4)**, we have $B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \equiv B(\nabla(\Phi, E_{w,w'}))$. Therefore $w \not\models B(\nabla(\Phi, E))$, a contradiction.

- Now we prove that $\llbracket B(\nabla(\Phi, E)) \rrbracket \supseteq \max(\llbracket B(E) \rrbracket, \geq_\Phi)$. Consider w, w' a pair of interpretations such that w is in $\max(\llbracket B(E) \rrbracket, \geq_\Phi)$ and $w' \models B(\nabla(\Phi, E))$. By **(ESF1)**, $w' \models B(E)$, and then $w \geq_\Phi w'$. Let $E_{w,w'}$ be an epistemic state satisfying $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$. By definition, $w \models B(\nabla(\Phi, E_{w,w'}))$. Since $B(\nabla(\Phi, E))$ is consistent with $B(E_{w,w'})$, by **(ESF3)** and **(ESF4)**, we have $B(\nabla(\Phi, E)) \wedge B(E_{w,w'}) \equiv B(\nabla(\Phi, E_{w,w'}))$. From this and the fact that $w \models B(\nabla(\Phi, E_{w,w'}))$, we have $w \models B(\nabla(\Phi, E))$.

Finally, from **(ESF2)** it follows straightforwardly that $\Phi \mapsto \geq_\Phi$ is really a basic assignment. Moreover, it is easy to see that this is the only basic assignment that satisfies (B-Rep). Indeed, any assignment satisfying (B-Rep) has to satisfy also the equivalence (6), so it is unique.

(If part) Assume that $\Phi \mapsto \geq_\Phi$ satisfies (B-Rep). The satisfaction of **(ESF1)** and **(ESF2)** is a straightforward consequence of (B-Rep) and by the fact that $\Phi \mapsto \geq_\Phi$ is a basic assignment. Thus, it remains to prove **(ESF3)** and **(ESF4)**.

(ESF3) Suppose we have a finite society of agents N in $\mathcal{F}^*(\mathcal{S})$, Φ an N -profile, and E, E', E'' three epistemic states such that $B(E) \equiv B(E') \wedge B(E'')$. We must show that

$$B(\nabla(\Phi, E')) \wedge B(E'') \vdash B(\nabla(\Phi, E))$$

If $B(\nabla(\Phi, E')) \wedge B(E'') \vdash \perp$ the above entailment is clear. Now suppose that $B(\nabla(\Phi, E')) \wedge B(E'') \not\vdash \perp$. Let w be an interpretation that satisfies $w \models B(\nabla(\Phi, E')) \wedge B(E'')$. Thus, from (B-Rep) we have that $w \models B(E') \wedge B(E'')$, and therefore $w \models B(E)$. Let w' be any interpretation such that $w' \models B(E)$; then $w' \models B(E')$. Since $w \models B(\nabla(\Phi, E'))$, by (B-Rep) we have $w \geq_\Phi w'$; and again by (B-Rep), we have $w \models B(\nabla(\Phi, E))$.

(ESF4) Let us consider N in $\mathcal{F}^*(\mathcal{S})$, an N -profile Φ and epistemic states E, E' and E'' such that $B(E) \equiv B(E') \wedge B(E'')$ and $B(\nabla(\Phi, E')) \wedge B(E'') \not\vdash \perp$. We want to prove that $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi, E')) \wedge B(E'')$. Let w be an interpretation such that $w \models B(\nabla(\Phi, E))$. Towards a contradiction, suppose that $w \not\models B(\nabla(\Phi, E')) \wedge B(E'')$. Let w' be a model of $B(\nabla(\Phi, E')) \wedge B(E'')$. Thus, by (B-Rep), $w' \models B(E') \wedge B(E'')$, and then $w' \models B(E)$. Since $w \models B(E')$, $w \not\models B(\nabla(\Phi, E'))$ and $w' \models B(\nabla(\Phi, E'))$, by (B-Rep), we have $w' >_\Phi w$. Again by (B-Rep), this means that $w \not\models B(\nabla(\Phi, E))$, a contradiction. \square

Proof of Proposition 2: Let ∇ be an ES basic fusion operator and let us consider $\Phi \mapsto \geq_\Phi$ the basic assignment given by theorem 1 associated to ∇ .

- (i) (Only if part) Let us suppose that ∇ satisfies **(ESF5)**. We want to show that $\Phi \mapsto \geq_\Phi$ satisfies **1**. Let j and k be a pair of agents in \mathcal{S} , and consider profiles E_j, E_k , such that $E_j \neq E_k$. We want to see that $\geq_{E_j} \neq \geq_{E_k}$. By **(ESF5)**, there exists an epistemic state E such that $B(\nabla(E_j, E)) \neq B(\nabla(E_k, E))$. From this, as a straightforward consequence of (B-Rep), we have that $\geq_{E_j} \neq \geq_{E_k}$.

(If part) If $\Phi \mapsto \geq_\Phi$ satisfies **1**, there exists a pair of interpretations w, w' such that $w \geq_{E_j} w'$ and $w' >_{E_k} w$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$, then $\max(\llbracket B(E) \rrbracket, \geq_{E_j}) \neq \max(\llbracket B(E) \rrbracket, \geq_{E_k})$. Thus, by (B-Rep) we have that $B(\nabla(E_j, E)) \neq B(\nabla(E_k, E))$.

- (ii) (Only if part) Assume that ∇ satisfies **(ESF6)**. Let us consider N in $\mathcal{F}^*(\mathcal{S})$ and Φ an N -profile such that $\bigwedge_{i \in N} B(E_i) \not\vdash \perp$. We want to show that $\max(\geq_\Phi) = \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket$.

- First we prove $\max(\geq_\phi) \subseteq \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket$. In order to do this, let w be an interpretation in $\max(\geq_\phi)$, and suppose, towards a contradiction, that $w \not\models \bigwedge_{i \in N} B(E_i)$. Let w' be a model of $\bigwedge_{i \in N} B(E_i)$. Note that $w \geq_\phi w'$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. Then, because of **(ESF6)**, we have that $B(\nabla(\Phi, E)) \equiv \bigwedge_{i \in N} B(E_i) \wedge B(E)$. Thus, $w \not\models B(\nabla(\Phi, E))$ and $w' \models B(\nabla(\Phi, E))$. Then, by (B-Rep), we have $w' >_\phi w$, a contradiction.
- Now we prove $\max(\geq_\phi) \supseteq \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket$. Let w be a model of $\bigwedge_{i \in N} B(E_i)$. We want to see that w is in $\max(\geq_\phi)$. Let w' be an interpretation and E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. Since w is a model of $\bigwedge_{i \in N} B(E_i) \wedge B(E)$, by **(ESF6)**, we have $w \models B(\nabla(\Phi, E))$. From this and (B-Rep), we have $w \geq_\phi w'$.

(If part) Assume that $\Phi \mapsto \geq_\phi$ satisfies Property 2. In order to show that ∇ satisfies **(ESF6)**, consider N in $\mathcal{F}^*(S)$, an N -profile Φ and an epistemic state E such that $\bigwedge_{i \in N} B(E_i) \wedge B(E) \not\models \perp$. In particular, $\bigwedge_{i \in N} B(E_i) \not\models \perp$. Thus, by Property 2, $\max(\geq_\phi) = \llbracket \bigwedge_{i \in N} B(E_i) \rrbracket$. Therefore, $\max(\llbracket B(E) \rrbracket, \geq_\phi) = \llbracket \bigwedge_{i \in N} B(E_i) \wedge B(E) \rrbracket$. Then, by (B-Rep), $B(\nabla(\Phi, E)) \equiv \bigwedge_{i \in N} B(E_i) \wedge B(E)$.

- (iii) (Only if part) We assume **(ESF7)**. In order to prove Property 3, consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile and w, w' interpretations such that $w \geq_{\Phi \upharpoonright_{N_1}} w'$ and $w \geq_{\Phi \upharpoonright_{N_2}} w'$. We want to show that $w \geq_\phi w'$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. By (B-Rep), we have that $w \models B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$. Then, by **(ESF7)**, we have $w \models B(\nabla(\Phi, E))$. Thus, by (B-Rep), $w \geq_\phi w'$.

(If part) Suppose that the assignment $\Phi \mapsto \geq_\phi$ satisfies Property 3. In order to prove that ∇ satisfies **(ESF7)**, consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile and E an epistemic state. If $B(\nabla(\Phi \upharpoonright_{N_1}, E))$ is inconsistent with $B(\nabla(\Phi \upharpoonright_{N_2}, E))$ the result follows straightforwardly. Now suppose that $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$ is consistent, and let w be a model of it. Let w' be any model of $B(E)$. By (B-Rep), we have $w \geq_{\Phi \upharpoonright_{N_1}} w'$ and $w \geq_{\Phi \upharpoonright_{N_2}} w'$. Thus, from Property 3, it follows that $w \geq_\phi w'$; and this is true for all w' such that $w' \models B(E)$, that is, $w \in \max(\llbracket B(E) \rrbracket, \geq_\phi)$. Then, by (B-Rep), we have that $w \models B(\nabla(\Phi, E))$.

- (iv) (Only if part) Assume that ∇ satisfies **(ESF8)**. In order to see that the assignment $\Phi \mapsto \geq_\phi$ satisfies Property 4, consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile, w, w' a pair of interpretations and E an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$, $w \geq_{\Phi \upharpoonright_{N_1}} w'$ and $w >_{\Phi \upharpoonright_{N_2}} w'$. Thus, by (B-Rep), w is the sole model of $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$. By **(ESF8)**, w is also the sole model of $B(\nabla(\Phi, E))$. From this, again by (B-Rep), we have $w >_\phi w'$.

(If part) Assume that the assignment $\Phi \mapsto \geq_\phi$ satisfies Property 4. Suppose, towards a contradiction, that ∇ does not satisfy **(ESF8)**. Thus, there exist N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile and w a model of $B(\nabla(\Phi, E))$ such that $w \not\models B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$ and $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$ is consistent. Note that $w \models B(E)$. Without loss of generality, suppose that $w \not\models B(\nabla(\Phi \upharpoonright_{N_2}, E))$, and assume w' is a model of $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$. From this, by **(ESF8)**, we have $w' \models B(\nabla(\Phi, E))$ and then, by (B-Rep), $w \geq_\phi w'$. Since $w' \models B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$ and $w \not\models B(\nabla(\Phi \upharpoonright_{N_2}, E))$, by (B-Rep), we have $w' \geq_{\Phi \upharpoonright_{N_1}} w$ and $w' >_{\Phi \upharpoonright_{N_2}} w$. Then, from Property 4, it follows that $w' >_\phi w$, a contradiction.

- (v) (Only if part) Assume that ∇ satisfies **(ESF8W)**. In order to see that the assignment $\Phi \mapsto \geq_\phi$ satisfies Property 4', consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile, and w, w' interpretations such that, $w >_{\Phi \upharpoonright_{N_1}} w'$ and $w >_{\Phi \upharpoonright_{N_2}} w'$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. By (B-Rep),

we have $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \vee B(\nabla(\Phi \upharpoonright_{N_2}, E))$ has a unique model, namely w . Thus, by **(ESF8W)**, w is also the unique model of $B(\nabla(\Phi, E))$. Therefore, by **(B-Rep)**, we have $w >_{\Phi} w'$.

(If part) Assume Property **4'**. In order to see that **(ESF8W)** holds, consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , Φ an N -profile and E an epistemic state such that $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E)) \not\models \perp$. Let w be a model of $B(\nabla(\Phi, E))$. We want to show that w is a model of $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \vee B(\nabla(\Phi \upharpoonright_{N_2}, E))$. Suppose that $w \not\models B(\nabla(\Phi \upharpoonright_{N_1}, E))$. Let w' be a model of $B(\nabla(\Phi \upharpoonright_{N_1}, E)) \wedge B(\nabla(\Phi \upharpoonright_{N_2}, E))$. Then, by **(B-Rep)**, we have that $w \geq_{\Phi} w'$, $w' >_{\Phi \upharpoonright_{N_1}} w$, $w' \geq_{\Phi \upharpoonright_{N_2}} w$. If $w' >_{\Phi \upharpoonright_{N_2}} w$, since $w' >_{\Phi \upharpoonright_{N_1}} w$, from Property **4'**, we have that $w' >_{\Phi} w$, a contradiction. Thus, necessarily $w \simeq_{\Phi \upharpoonright_{N_2}} w'$, and, by **(B-Rep)**, we have $w \models B(\nabla(\Phi \upharpoonright_{N_2}, E))$. Therefore $w \models B(\nabla(\Phi \upharpoonright_{N_1}, E)) \vee B(\nabla(\Phi \upharpoonright_{N_2}, E))$, as desired. \square

Proof of Theorem 4: First of all, note that there are four interpretations and therefore fifteen elements in $*/\equiv$. Since ∇ is a basic operator, the assignment restraint to profiles of size one has at most fifteen elements in its image. That is, the function $\varphi \mapsto \geq_{\varphi}$ has at most fifteen images. We suppose that the Maximality Condition holds, that is, $\max(\geq_{\varphi}) = \llbracket \varphi \rrbracket$. Table 2 will be useful for our combinatorial analysis.

Table 2. Types of preorders associated to formulas according to the number of their models

# of models	# of formulas φ	possible types of \geq_{φ}
4	1	$\bullet \bullet \bullet \bullet$
3	4	$\bullet \bullet \bullet$ \bullet
2	6	$\bullet \bullet$ $\bullet \bullet$ \bullet
1	4	\bullet \bullet \bullet \bullet

By Observation 1, all the preorders \geq_{φ} have to cover all the shapes of the following types:

$$S_1 = \bullet \bullet \bullet \quad S_2 = \begin{array}{c} \bullet \bullet \\ \bullet \end{array} \quad S_3 = \begin{array}{c} \bullet \\ \bullet \bullet \end{array} \quad S_4 = \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$$

where the points represent 3 arbitrary models taken among the four models we are considering. It is clear that the first two shapes, *i.e.* the shapes of the form S_1 and S_2 , can be covered by the images \geq_{φ} of the formulas of 4 and 3 models (actually, the image of formulas of 3 models are enough to cover all the shapes of the type S_1 and S_2). It is also clear that the images of formulas of 4 and 3 models can not cover any shape of types S_3 and S_4 . Thus, the problem is now reduced to knowing if it is possible to cover all the shapes of types S_3 and S_4 with the images of formulas having 2 models or 1 model. We claim that this is impossible.

First we consider all the types of total preorders which can be the image by the assignment of formulas having 2 models (D_1 and D_2) or 1 model (T_1, \dots, T_4) in the following way:

$$D_1 = \begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} \quad D_2 = \begin{array}{c} \bullet \bullet \\ \bullet \\ \bullet \end{array} \quad T_1 = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \quad T_2 = \begin{array}{c} \bullet \\ \bullet \\ \bullet \bullet \end{array} \quad T_3 = \begin{array}{c} \bullet \\ \bullet \bullet \\ \bullet \end{array} \quad T_4 = \begin{array}{c} \bullet \\ \bullet \bullet \bullet \end{array}$$

In order to see the previously mentioned impossibility, note that there are 24 possible patterns for the shapes of type S_4 (the number of ways to select 3 elements of 4, that is $\binom{4}{3}$, multiplied by the number of all possible orders with the 3 selected elements, that is $3!$). A similar analysis shows that the number of possible patterns for the shapes of type S_3 is 12 (the number of ways to select 2 elements of 4, that is $\binom{4}{2}$, multiplied by the number of ways to select 1 among the 2 remainder elements, that is 2).

It is easy to see that the following claims hold:

- A total pre-order of the type T_1 covers 4 patterns of type S_4 and 0 pattern of type S_3 .
- A total pre-order of the type T_2 covers 2 patterns of type S_4 and 2 patterns of type S_3 .
- A total pre-order of the type T_3 covers 2 patterns of type S_4 and 1 pattern of type S_3 .
- A total pre-order of the type T_4 covers 0 patterns of type S_4 and 3 patterns of type S_3 .
- A total pre-order of the type D_2 covers 2 patterns of type S_4 and 0 pattern of type S_3 .
- A total pre-order of the type D_1 covers 0 patterns of type S_4 and 2 patterns of type S_3 .

Now we analyse where could the four formulas having one model be mapped under the assignment representing ∇ . This information is represented by a vector (n_1, n_2, n_3, n_4) of integers greater than or equal to zero, where n_i is the number of formulas with one model having image a pre-order of type T_i . Note that $\sum_{i=1}^4 n_i = 4$. We use the same kind of vectorial representation to see where could the six formulas having two models be mapped, that is, a vector (m_1, m_2) of integers greater than or equal to zero, where m_i is the number the formulas with two models having image a pre-order of type D_i . Note that $m_1 + m_2 = 6$.

In Table 3 we show the distributions that can cover the 24 patterns for the shapes of type S_4 and in the second column appears the maximal number of S_3 patterns covered by each distribution:

Table 3. Table of distributions covering the 24 shapes of type S_4 and maximal number of shapes S_3 covered by them

Distributions	Maximal number of patterns of type S_3 covered
$\{(4, 0, 0, 0), (0, 6)\}$	0
$\{(4, 0, 0, 0), (1, 5)\}$	2
$\{(4, 0, 0, 0), (2, 4)\}$	4
$\{(3, 1, 0, 0), (0, 6)\}$	2
$\{(3, 1, 0, 0), (1, 5)\}$	4
$\{(3, 0, 1, 0), (0, 6)\}$	1
$\{(3, 0, 1, 0), (1, 5)\}$	3
$\{(3, 0, 0, 1), (0, 6)\}$	3

Therefore, each distribution covering the 24 patterns of type S_4 , covers strictly less than 12 patterns of type S_3 . This observation finishes the proof. \square

Proof of Proposition 3: Assume that ∇ is an ES basic fusion operator and let $\Phi \mapsto \succeq_\Phi$ be the basic assignment associated to ∇ by Theorem 1. Let i be an agent and E_i be an i -profile and w, w', w'' be three different interpretations in \mathcal{W}_Φ .

- (i) If $w' \geq_{E_i} w''$, by **(ESF-SD)**, given j in S there exists a j -profile E_j satisfying that $B(\nabla(E_j, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_j, E_{w',w''})) \equiv B(\nabla(E_i, E_{w',w''}))$ (the last equivalence uses (iii) of **(ESF-SD)** if $w' \simeq_{E_i} w''$ and uses (iv) of **(ESF-SD)** if $w' >_{E_i} w''$). From this and (B-Rep), we get $w >_{E_j} w'$. Moreover, since $w' \geq_{E_i} w''$ and $B(\nabla(E_j, E_{w',w''})) \equiv B(\nabla(E_i, E_{w',w''}))$, by (B-Rep), we have that $w' \geq_{E_j} w''$. Thus, by transitivity, $w >_{E_j} w''$. Therefore, by (B-Rep), $B(\nabla(E_j, E_{w,w'})) \equiv \varphi_w$. If $w'' \geq_{E_i} w'$, the proof is analogous, interchanging the roles of w' and w'' .
- (ii) If $w \geq_{E_i} w''$, by **(ESF-SD)**, given k in S there exists a k -profile E_k satisfying that $B(\nabla(E_k, E_{w',w''})) \equiv \varphi_{w''}$ and $B(\nabla(E_k, E_{w,w'})) \equiv B(\nabla(E_i, E_{w,w'}))$ (the last equivalence uses (iii) of **(ESF-SD)** if $w \simeq_{E_i} w''$ and uses (iv) of **(ESF-SD)** if $w >_{E_i} w''$). Thus, since $w \geq_{E_i} w''$, $B(\nabla(E_k, E_{w,w'})) \equiv B(\nabla(E_i, E_{w,w'}))$ and $B(\nabla(E_k, E_{w',w''})) \equiv \varphi_{w''}$, from (B-Rep), we have that $w \geq_{E_k} w''$ and $w'' >_{E_k} w'$. Then $w >_{E_k} w'$, that is, by (B-Rep), $B(\nabla(E_k, E_{w,w'})) \equiv \varphi_w$. If $w'' \geq_E w$, the proof is analogous, interchanging the roles of w and w'' . \square

Proof of Proposition 4: Suppose that ∇ satisfies **(ESF2)**, **(ESF7)** and **(ESF8)**. In order to show **(ESF-U)**, we proceed by induction on the length of finite societies: we will show that for every N in $\mathcal{F}^*(S)$, N satisfies the following property:

- (U) For all N -profile Φ and every E in \mathcal{E} , if $E_i = E_j$, for all i, j in N , then $B(\nabla(\Phi, E)) \equiv B(\nabla(E_i, E))$, for all i in N .

If N has only one agent, (U) holds trivially. Now, let n be a positive integer, with $n > 1$, and suppose that for all D in $\mathcal{F}^*(S)$, formed by a number of agents less than n , it satisfies (U). Consider N in $\mathcal{F}^*(S)$ with n agents and let us see that N also satisfies (U). Consider an N -profile Φ such that $E_i = E_j$, for every couple i, j in N , and let E be an epistemic state. Consider $D = N \setminus \{k\}$, for some k in N . Thus, if i is any agent in D , from (U) we have that $B(\nabla(\Phi \upharpoonright_D, E)) \equiv B(\nabla(E_i, E))$. Moreover, since $E_k = E_i$, by **(ESF2)** we have that $B(\nabla(E_k, E)) \equiv B(\nabla(E_i, E))$. Thus, $B(\nabla(\Phi \upharpoonright_D, E)) \wedge B(\nabla(E_k, E)) \equiv B(\nabla(E_i, E))$, for all i in N . From the last equivalence, the result follows using **(ESF7)** and **(ESF8)**. \square

Proof of Proposition 5: Let us suppose N in $\mathcal{F}^*(S)$ and an N -profile Φ such that $E_i = E_j$ for any couple i, j in N .

- (i) \Rightarrow (ii) Trivial.
- (ii) \Rightarrow (iii) Let w, w' be a pair of interpretations and consider E an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. Since $E_i = E_j$, for any pair of agents i, j in N , from (ii) we have that $B(\nabla(\Phi, E)) \equiv B(\nabla(E_i, E))$, for all i in N . Thus, by (B-Rep), we have $w \geq_\Phi w'$ iff $w \geq_{E_i} w'$, for all i in N .
- (iii) \Rightarrow (i) Let E be an epistemic state. Since $E_i = E_j$, for any pair of agents i, j in N , by Property (u) we have $\geq_\Phi = \geq_{E_i}$ for all i in N . From this, by (B-Rep), $B(\nabla(\Phi, E)) \equiv B(\nabla(E_i, E))$, for all i in N . \square

Proof of Proposition 6: Let ∇ be an ES combination operator, and suppose that it satisfies **(ESF7)** and **(ESF8W)**. In order to show that **(ESF-P)** holds, we will proceed by induction on the length of finite societies. Indeed, we will show that, for all N in $\mathcal{F}^*(S)$, N satisfies the following:

- (P) For any N -profile Φ , any pair of epistemic states E, E' in \mathcal{E} , if $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in N , and $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.

If N has a sole agent, the result is trivial. Now, assume $n > 1$ a positive integer and suppose that, for all D in $\mathcal{F}^*(S)$ with less than n agents, D satisfies (P). Consider N in $\mathcal{F}^*(S)$ with n agents and let us show that N also satisfies (P). Let Φ be an N -profile and E, E' a pair of epistemic states such that $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in N and $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$. Let j be an agent in N and let us consider $D = N \setminus \{j\}$. Thus,

$B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in D and $\bigwedge_{i \in D} B(\nabla(E_i, E)) \not\vdash \perp$. From this and the induction hypothesis (i.e. **(P)** for the profile $\Phi \upharpoonright_D$), we have that $B(\nabla(\Phi \upharpoonright_D, E)) \wedge B(E') \vdash \perp$. Moreover, by **(ESF7)**, it is easy to see that $\bigwedge_{i \in D} B(\nabla(E_i, E)) \vdash B(\nabla(\Phi \upharpoonright_D, E))$. Then, $\bigwedge_{i \in N} B(\nabla(E_i, E)) \vdash B(\nabla(\Phi \upharpoonright_D, E)) \wedge B(\nabla(E_j, E))$, and since $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$ then $B(\nabla(\Phi \upharpoonright_D, E)) \wedge B(\nabla(E_j, E)) \not\vdash \perp$. Therefore, by **(ESF8W)**, we have that $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi \upharpoonright_D, E)) \vee B(\nabla(E_j, E))$. From this and the fact that $[B(\nabla(\Phi \upharpoonright_D, E)) \vee B(\nabla(E_j, E))] \wedge B(E') \vdash \perp$, we get $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$, as desired. \square

Proof of Proposition 7: Let N be in $\mathcal{F}^*(S)$, Φ be an N -profile and E, E' be a pair of epistemic states.

- (i) \Rightarrow (ii) Suppose that $B(E)$ has at most two models and assume that $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in N . We want to show that $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$. If $B(E) \wedge B(E') \vdash \perp$, the result follows straightforwardly from **(ESF1)**. Now suppose that $B(E) \wedge B(E') \not\vdash \perp$ and let w' be a model of $B(E) \wedge B(E')$. If w' were the sole model of $B(E)$, then, by **(ESF1)**, w' were also a model of $B(\nabla(E_i, E))$, for each i in N . Thus, if we consider j in N , we have that $B(\nabla(E_j, E)) \wedge B(E') \not\vdash \perp$, a contradiction. Thus, $B(E)$ has two models. Let w be the other model of $B(E)$, in particular $w \neq w'$. Since $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, we have, by **(ESF1)**, that w is the sole model of $B(\nabla(E_i, E))$, for all i in N . Therefore, $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$. Then, by **(ESF-P)**, $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.
- (ii) \Rightarrow (iii) Let w, w' be a pair of interpretations which satisfies $w >_{E_i} w'$, for all i in N , and assume that E, E' are epistemic states which satisfy $\llbracket B(E) \rrbracket = \{w, w'\}$ and $\llbracket B(E') \rrbracket = \{w'\}$. Thus, by (B-Rep) we have that for all i in N , $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$. From this and the assumption that (ii) holds, we get $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$, and then, necessarily, $\llbracket B(\nabla(\Phi, E)) \rrbracket = \{w\}$. By (B-Rep), this last equality entails $w >_\Phi w'$.
- (iii) \Rightarrow (i) Let us suppose that $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in N , and $\bigwedge_{i \in N} B(\nabla(E_i, E))$ is consistent. Towards a contradiction, suppose that $B(\nabla(\Phi, E)) \wedge B(E') \not\vdash \perp$. Let w, w' be interpretations such that $w \models \bigwedge_{i \in N} B(\nabla(E_i, E))$ and $w' \models B(\nabla(\Phi, E)) \wedge B(E')$. By **(ESF1)**, we have that both w and w' are models of $B(E)$. Moreover, note that if i is an agent in N , $w' \not\models B(\nabla(E_i, E))$, because $w' \models B(E')$ and $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$. From this and (B-Rep), it follows that $w >_{E_i} w'$, for all i in N . Hence, by Property **(p)** we have $w >_\Phi w'$. From this and (B-Rep) again, it follows that $w' \not\models B(\nabla(\Phi, E))$, a contradiction. \square

Proof of Proposition 8: Let N be a finite society of agents in S and Φ, Φ' be a pair of N -profiles.

- (i) \Rightarrow (ii) Let E be an epistemic state such that $|B(E)| \leq 2$ and assume that $B(\nabla(E_i, E)) \equiv B(\nabla(E'_i, E))$, for each agent i in N . We want to show that $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$. By **(ESF-I)**, it is enough to prove that, for every epistemic state E' in \mathcal{E} , if $B(E') \vdash B(E)$, then $B(\nabla(E_i, E')) \equiv B(\nabla(E'_i, E'))$, for each agent i in N . Thus, suppose that E' is an epistemic state such that $B(E') \vdash B(E)$ and let i be any agent in N . If $B(E')$ has exactly one model, the result follows directly from **(ESF1)**. Otherwise, $B(E') \equiv B(E)$. By **(ESF2)** we have that $B(\nabla(E_i, E)) \equiv B(\nabla(E_i, E'))$ and $B(\nabla(E'_i, E)) \equiv B(\nabla(E'_i, E'))$. From these equivalences and the fact that $B(\nabla(E_i, E)) \equiv B(\nabla(E'_i, E))$, we have $B(\nabla(E_i, E')) \equiv B(\nabla(E'_i, E'))$.
- (ii) \Rightarrow (iii) Assume that (ii) holds. Let w, w' be a pair of models and suppose that for each i in N , $\geq_{E_i} \upharpoonright_{\{w, w'\}} = \geq_{E'_i} \upharpoonright_{\{w, w'\}}$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. From (B-Rep), we have $B(\nabla(E_i, E)) \equiv B(\nabla(E'_i, E))$, for all i in N . Hence, by the assumption that (ii) holds, $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$. Finally, by (B-Rep), $\geq_\Phi \upharpoonright_{\{w, w'\}} = \geq_{\Phi'} \upharpoonright_{\{w, w'\}}$.
- (iii) \Rightarrow (i) Let E be an epistemic state. Suppose that for every epistemic state E' , with $B(E') \vdash B(E)$, we have $B(\nabla(E_i, E')) \equiv B(\nabla(E'_i, E'))$, for each agent i in N . We have to show that $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$. In order to do this, it is enough to prove that $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi', E))$ (the proof of the converse is similar). Let w be a model of $B(\nabla(\Phi, E))$ and, towards a contradiction, suppose that $w \not\models B(\nabla(\Phi', E))$. Let w' be

a model of $B(\nabla(\Phi', E))$ and E' be an epistemic state with $\llbracket B(E') \rrbracket = \{w, w'\}$. Thus, $B(E') \vdash B(E)$. Then, for any agent i in N we have $B(\nabla(E_i, E')) \equiv B(\nabla(E'_i, E'))$. Then, by (B-Rep), $\geq_{E_i} \upharpoonright_{\{w, w'\}} = \geq_{E'_i} \upharpoonright_{\{w, w'\}}$, for all i in N . Therefore, from Property (ind) it follows that $\geq_\Phi \upharpoonright_{\{w, w'\}} = \geq_{\Phi'} \upharpoonright_{\{w, w'\}}$. From this and (B-Rep), we have $B(\nabla(\Phi, E')) \equiv B(\nabla(\Phi', E'))$. Moreover, since $B(E') \equiv B(E) \wedge B(E')$, $B(\nabla(\Phi, E)) \wedge B(E') \not\vdash \perp$ and $B(\nabla(\Phi', E)) \wedge B(E') \not\vdash \perp$, by (ESF3) and (ESF4), we have $B(\nabla(\Phi, E')) \equiv B(\nabla(\Phi, E)) \wedge B(E')$ and $B(\nabla(\Phi', E')) \equiv B(\nabla(\Phi', E)) \wedge B(E')$. Note that, since w is a model of $B(\nabla(\Phi, E)) \wedge B(E')$, then using the last two equivalences, we get that w is also a model of $B(\nabla(\Phi', E)) \wedge B(E')$, a contradiction. \square

Proof of Proposition 9: Let ∇ be an ES basic fusion operator which satisfies (ESF-U) and (ESF-I). We want to show that it satisfies (ESF-P). By Proposition 7, it is enough to prove that Property (ii) of this proposition holds. In order to do this, take N in $\mathcal{F}^*(S)$, Φ an N -profile, and let E, E' be a pair of epistemic states, such that $B(E)$ has at most two models. Suppose that, for all i in N , $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$. Towards a contradiction, suppose that $B(\nabla(\Phi, E)) \wedge B(E') \not\vdash \perp$. Let w' be a model of $B(\nabla(\Phi, E)) \wedge B(E')$. Note that for every $i \in N$,

since $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, by (ESF1), there exists w , a model of $B(E)$, which is, necessarily, the unique model of $B(\nabla(E_i, E))$. Let us fix an agent j in N and consider the N -profile Φ' , the entries of which are all equal to E_j , i.e. for all $i \in N$, $E'_i = E_j$. Thus, $B(\nabla(E_i, E)) \equiv B(\nabla(E'_i, E))$ for all i in N . Thus, by (ESF-I) and Proposition 8, we have $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$. Moreover, by (ESF-U), $B(\nabla(\Phi', E)) \equiv B(\nabla(E_j, E))$. Hence, $B(\nabla(\Phi, E)) \wedge B(E') \equiv B(\nabla(E_j, E)) \wedge B(E')$, and then $B(\nabla(E_j, E)) \wedge B(E') \not\vdash \perp$, a contradiction. \square

Proof of Proposition 10: Let N be an element of $\mathcal{F}^*(S)$.

(i) \Rightarrow (ii) Straightforward.

(ii) \Rightarrow (iii) Assume that d_N is the agent in N which satisfies (ii). Consider Φ an N -profile and w, w' a pair of interpretations such that $w \succ_{E_{d_N}} w'$. Let E be an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$. Note that, by (B-Rep), w is the sole model of $B(\nabla(E_{d_N}, E))$. Thus, since $B(\nabla(\Phi, E)) \vdash B(\nabla(E_{d_N}, E))$, we have that w is also the sole model of $B(\nabla(\Phi, E))$. Therefore, from (B-Rep), it follows that $w \succ_\Phi w'$.

(iii) \Rightarrow (i) Suppose that $\Phi \mapsto \geq_\Phi$ satisfies (d), and let d_N be the agent in N satisfying Property (d). Let Φ be an N -profile, E be an epistemic state. Suppose, towards a contradiction, that $B(\nabla(\Phi, E)) \not\vdash B(\nabla(E_{d_N}, E))$. Thus, let us choose w' , a model of $B(\nabla(\Phi, E))$, such that $w' \not\models B(\nabla(E_{d_N}, E))$. Then, by (B-Rep), there exists w , a model of $B(E)$, such that $w \succ_{E_{d_N}} w'$. Then, by Property (d), $w \succ_\Phi w'$. By (B-Rep) again, we have $w' \not\models B(\nabla(\Phi, E))$, a contradiction. \square

Proof of Proposition 11: Let ∇ be an ES combination operator that satisfies (ESF-D). Let N be a finite society of agents in S , Φ be an N -profile and E, E' be a pair of epistemic states in S such that $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ for all i in N , and $\bigwedge_{i \in N} B(\nabla(E_i, E)) \not\vdash \perp$. By (ESF-D), there exists d_N in N with $B(\nabla(\Phi, E)) \vdash B(\nabla(E_{d_N}, E))$. Thus, $B(\nabla(\Phi, E)) \wedge B(E') \vdash B(\nabla(E_{d_N}, E)) \wedge B(E')$. Therefore $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$, as desired. \square

Proof of Proposition 12: Suppose that $D = \{d\}$ for an agent d in N . Let us see that d is an N -dictator, with respect to ∇ . In order to see this, consider an N -profile Φ and an epistemic state E . Towards a contradiction, suppose that $B(\nabla(\Phi, E)) \not\vdash B(\nabla(E_d, E))$. Let w be a model of $B(\nabla(\Phi, E))$ such that $w \not\models B(\nabla(E_d, E))$, and consider E_w an epistemic state with $\llbracket B(E_w) \rrbracket = \{w\}$. Thus, $B(\nabla(E_d, E)) \wedge B(E_w) \vdash \perp$, and since $\{d\}$ is decisive, we have that $B(\nabla(\Phi, E)) \wedge B(E_w) \vdash \perp$. Therefore, $w \not\models B(\nabla(\Phi, E))$, a contradiction. \square

Proof of Proposition 14: Assume that $E_{w, w'} D^\nabla E_{w'}$, for a pair of epistemic states $E_{w, w'}$ and $E_{w'}$ with $\llbracket B(E_{w, w'}) \rrbracket = \{w, w'\}$ and $\llbracket B(E_{w'}) \rrbracket = \{w'\}$. We want to show that, if w'' is an interpretation different from w and w' , we have $E_{w, w''} D^{\nabla^*} E_{w''}$, for any epistemic states $E_{w, w''}$ and $E_{w''}$ such that $\llbracket B(E_{w, w''}) \rrbracket = \{w, w''\}$ and $\llbracket B(E_{w''}) \rrbracket = \{w''\}$. In order to do this, suppose that Φ is an N -profile such that $B(\nabla(E_i, E_{w, w''})) \wedge B(E_{w''}) \vdash \perp$, for all i in D , and consider $E_{w', w''}$ an epistemic state such that $\llbracket B(E_{w', w''}) \rrbracket = \{w', w''\}$. Thus, since ∇ satisfies (ESF-SD), by Proposition 3, there exists an N -profile Φ' which satisfies the following:

- (i) For all i in D :
 - $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_w$, and
 - $B(\nabla(E'_i, E_{w',w''})) \equiv \varphi_{w'}$
- (ii) For all j in $N \setminus D$:
 - $B(\nabla(E'_j, E_{w,w'})) \equiv B(\nabla(E'_j, E_{w',w''})) \equiv \varphi_{w'}$, and
 - $B(\nabla(E'_j, E_{w,w''})) \equiv B(\nabla(E_j, E_{w,w''}))$

Since $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_w$ for all i in D , and $B(\nabla(E'_j, E_{w,w'})) \equiv \varphi_{w'}$ for all j in $N \setminus D$, then:

- $B(\nabla(E'_i, E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$, for all i in D
- $B(\nabla(E'_j, E_{w,w'})) \equiv B(E_{w'})$, for all j in $N \setminus D$
- $\bigwedge_{i \in D} B(\nabla(E'_i, E_{w,w'})) \not\vdash \perp$

Then, because of $E_{w,w'} D^\nabla E_{w'}$, we have that $B(\nabla(\Phi', E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$. Thus, by **(ESF1)**, we have

$$B(\nabla(\Phi', E_{w,w'})) \equiv \varphi_w \quad (11)$$

Moreover, for all i in N , $B(\nabla(E'_i, E_{w',w''})) \equiv \varphi_{w'}$, thus $B(\nabla(E'_i, E_{w',w''})) \wedge B(E_{w''}) \vdash \perp$. Then, by **(ESF-P)**, we have $B(\nabla(\Phi', E_{w',w''})) \wedge B(E_{w''}) \vdash \perp$. From this and **(ESF1)** we get

$$B(\nabla(\Phi', E_{w',w''})) \equiv \varphi_{w'} \quad (12)$$

Thus, by the equivalences (11) and (12), from (B-Rep) and the transitivity of $\geq_{\Phi'}$ it follows that

$$B(\nabla(\Phi', E_{w,w''})) \equiv \varphi_w \quad (13)$$

Now, given i in D , since $B(\nabla(E_i, E_{w,w''})) \wedge B(E_{w''}) \vdash \perp$, by **(ESF1)** we have that $B(\nabla(E_i, E_{w,w''})) \equiv \varphi_w$. Moreover, $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E'_i, E_{w',w''})) \equiv \varphi_{w'}$ together imply $B(\nabla(E'_i, E_{w,w''})) \equiv \varphi_w$. Therefore, for all i in D , $B(\nabla(E_i, E_{w,w''})) \equiv B(\nabla(E'_i, E_{w,w''}))$. From this and the fact that $B(\nabla(E'_j, E_{w,w''})) \equiv B(\nabla(E_j, E_{w,w''}))$ for all j in $N \setminus D$, it follows that $B(\nabla(E_i, E_{w,w''})) \equiv B(\nabla(E'_i, E_{w,w''}))$ for all i in N . Hence, by **(ESF-I)** and Proposition 8, we have that $B(\nabla(\Phi, E_{w,w''})) \equiv B(\nabla(\Phi', E_{w,w''}))$. From this and Equivalence (13), it follows $B(\nabla(\Phi, E_{w,w''})) \wedge B(E_{w''}) \vdash \perp$. \square

Proof of Proposition 15: Let w'' be an interpretation, with $w'' \not\models B(E_{w,w'})$, and $E_{w',w''}$ an epistemic state such that $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$. Suppose that $E_{w,w'} D^\nabla E_{w'}$ and let us see that $E_{w',w''} D^{\nabla^*} E_{w'}$. In order to do that, take an N -profile Φ such that $B(\nabla(E_i, E_{w,w''})) \wedge B(E_{w'}) \vdash \perp$, for all i in D . Let $E_{w,w''}$ be an epistemic state such that $\llbracket B(E_{w,w''}) \rrbracket = \{w, w''\}$. By **(ESF-SD)** and Proposition 3, there exists an N -profile Φ' which satisfies the following:

- (i) $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E'_i, E_{w,w''})) \equiv \varphi_{w''}$, for all i in D .
- (ii) $B(\nabla(E'_j, E_{w,w'})) \equiv \varphi_{w'}$, $B(\nabla(E'_j, E_{w,w''})) \equiv \varphi_{w''}$ and $B(\nabla(E'_j, E_{w',w''})) \equiv B(\nabla(E_j, E_{w',w''}))$, for all j in $N \setminus D$.

From these statements it follows that:

- $B(\nabla(E'_i, E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$, for all i in D
- $B(\nabla(E'_j, E_{w,w'})) \equiv B(E_{w'})$, for all j in $N \setminus D$
- $\bigwedge_{i \in D} B(\nabla(E'_i, E_{w,w'})) \not\vdash \perp$

Thus, since $E_{w,w'} D_{E_{w'}}^\nabla$, we have $B(\nabla(\Phi', E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$. Hence, by **(ESF1)**, it follows

$$B(\nabla(\Phi', E_{w,w'})) \equiv \varphi_w \quad (14)$$

Note that $B(\nabla(E'_i, E_{w,w'})) \wedge B(E_w) \vdash \perp$, for all i in N . Hence, by **(ESF-P)**, we have $B(\nabla(\Phi', E_{w,w'})) \wedge B(E_w) \vdash \perp$. Thus, by **(ESF1)**, we have:

$$B(\nabla(\Phi', E_{w,w'})) \equiv \varphi_{w'} \quad (15)$$

From the equivalences (14) and (15), (B-Rep) and transitivity of $\geq_{\Phi'}$, it follows that $B(\nabla(\Phi', E_{w,w'})) \equiv \varphi_{w''}$. Now, if i is an agent in D , since $B(\nabla(E_i, E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$, **(ESF1)** implies that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_{w''}$. Moreover, $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_{w''}$ together entail $B(\nabla(E'_i, E_{w,w'})) \equiv \varphi_{w''}$. Thus, for all i in D , $B(\nabla(E_i, E_{w,w'})) \equiv B(\nabla(E'_i, E_{w,w'}))$. Moreover, $B(\nabla(E_i, E_{w,w'})) \equiv B(\nabla(E'_i, E_{w,w'}))$, for all i in N . Therefore, by **(ESF-I)**, we have that $B(\nabla(\Phi, E_{w,w'})) \equiv B(\nabla(\Phi', E_{w,w'}))$. Then, by the equivalence (15), it follows that $B(\nabla(\Phi, E_{w,w'})) \wedge B(E_{w'}) \vdash \perp$. \square

Proof of Theorem 5: Consider w, w' a pair of interpretations and assume that $E_{w,w'}$ and E_w are epistemic states which satisfy $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_w) \rrbracket = \{w'\}$ and $E_{w,w'} D^\nabla E_w$. We want to show that D is decisive, with respect to ∇ . In order to do this, by Proposition 13, it is enough to show that for each pair of epistemic states, E, E' , such that $B(E)$ has at most two models and $B(E')$ has exactly one model, we have that $ED^\nabla E'$. First, let w'' be an interpretation different from w and w' . We prove that for any pair of different interpretations w_1, w_2 in $\{w, w', w''\}$ we have that $E_{w_1, w_2} D^\nabla E_{w_2}$. For that purpose, we note the following:

- Since $E_{w,w'} D^\nabla E_w$, by Proposition 14, it follows $E_{w,w'} D^\nabla E_{w''}$.
- Since $E_{w,w'} D^\nabla E_w$, by Proposition 15, we have $E_{w,w'} D^\nabla E_{w'}$.
- Since $E_{w,w'} D^\nabla E_{w''}$, by Observation 2, it follows $E_{w,w'} D^\nabla E_{w'}$. Thus, by Proposition 14, we have $E_{w,w'} D^\nabla E_{w'}$.
- Since $E_{w,w'} D^\nabla E_{w'}$, by Observation 2, it follows $E_{w,w'} D^\nabla E_w$. Thus, by Proposition 14, we have $E_{w,w'} D^\nabla E_w$.
- Since $E_{w,w'} D^\nabla E_w$, by Proposition 15, it follows $E_{w,w'} D^\nabla E_{w''}$.
- Since $E_{w,w'} D^\nabla E_{w''}$, by Observation 2, we have $E_{w,w'} D^\nabla E_{w'}$. Then, by Proposition 14, we have $E_{w,w'} D^\nabla E_w$.

Now, let w_1, w_2 be a pair of different interpretations. If either w_1 or w_2 , is in $\{w, w'\}$, from the above statements we have $E_{w_1, w_2} D^\nabla E_{w_2}$. Hence, we suppose that both w_1 and w_2 are not elements of $\{w, w'\}$. Since $E_{w,w'} D^\nabla E_w$, by Proposition 15, we have $E_{w_1, w'} D^\nabla E_{w'}$, and thus $E_{w_1, w'} D^\nabla E_w$. From this, by Proposition 14, $E_{w_1, w_2} D^\nabla E_{w_2}$. \square

Proof of Theorem 6: Let N be a finite society of agents in \mathcal{S} . Define the following set:

$$\mathfrak{X} = \{X \subseteq N : X \text{ is decisive with respect to } \nabla\}.$$

Since we have **(ESF-P)**, by Observation 2, N is decisive. Thus, \mathfrak{X} is non empty. Let D be an element of \mathfrak{X} having minimal cardinality. Since D is decisive then $D \neq \emptyset$, as we noted in Observation 2. Thus, by the same observation, it is enough to see that D has a single model in order to prove that ∇ is dictatorial. Towards a contradiction, suppose that D has at least two agents. Let i be an agent in D . Take three different interpretations w, w', w'' . Let $E_{w,w'}, E_{w,w''}, E_{w',w''}, E_w$ and $E_{w''}$ be epistemic states such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$, $\llbracket B(E_{w,w''}) \rrbracket = \{w, w''\}$, $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$, $\llbracket B(E_w) \rrbracket = \{w\}$ and $\llbracket B(E_{w''}) \rrbracket = \{w''\}$. By **(ESF-SD)**, there exists an N -profile Φ which satisfies the following:

- $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_i, E_{w,w''})) \equiv \varphi_{w'}$.
- $B(\nabla(E_j, E_{w,w''})) \equiv \varphi_{w''}$ and $B(\nabla(E_j, E_{w',w''})) \equiv \varphi_{w'}$, for all j in $D \setminus \{i\}$.
- $B(\nabla(E_k, E_{w,w''})) \equiv \varphi_w$ and $B(\nabla(E_k, E_{w,w''})) \equiv \varphi_{w''}$, for all k in $N \setminus D$.

Note that, if j is an agent in D , then $B(\nabla(E_j, E_{w',w''})) \wedge B(E_{w''}) \vdash \perp$. Thus, since D is decisive, we have in particular $E_{w',w''} D^{\nabla} E_{w''}$. Therefore

$$B(\nabla(\Phi, E_{w',w''})) \wedge B(E_{w''}) \vdash \perp \quad (16)$$

We claim that $w \models B(\nabla(\Phi, E_{w,w'}))$. Towards a contradiction, suppose that $B(\nabla(\Phi, E_{w,w'})) \wedge B(E_w) \vdash \perp$. By Proposition 3, there exists an N -profile Φ' such that

- $B(\nabla(E'_j, E_{w,w'})) \wedge B(E_w) \vdash \perp$, for all j in $D \setminus \{i\}$
- $B(\nabla(E'_k, E_{w,w'})) \equiv B(E_w)$, for all k in $[N \setminus D] \cup \{i\}$

By **(ESF1)**, we have that $\bigwedge_{j \in D \setminus \{i\}} B(\nabla(E'_j, E_{w,w'})) \not\models \perp$, actually $B(\nabla(E'_j, E_{w,w'})) \equiv \varphi_{w'}$, for all $j \in D \setminus \{i\}$. Moreover, for all j in $D \setminus \{i\}$, we have $B(\nabla(E_j, E_{w,w''})) \equiv \varphi_{w''}$ and $B(\nabla(E_j, E_{w',w''})) \equiv \varphi_{w'}$, and so $B(\nabla(E_j, E_{w,w'})) \equiv \varphi_{w'}$. Thus, $B(\nabla(E_j, E_{w,w'})) \equiv B(\nabla(E'_j, E_{w,w'}))$, for all j in $D \setminus \{i\}$. Hence, $B(\nabla(E_j, E_{w,w'})) \equiv B(\nabla(E'_j, E_{w,w'}))$ for all j in N . From this, by **(ESF-I)** and Proposition 8, it follows that $B(\nabla(\Phi, E_{w,w'})) \equiv B(\nabla(\Phi', E_{w,w'}))$. Thus, $B(\nabla(\Phi', E_{w,w'})) \wedge B(E_w) \vdash \perp$. Hence, $E_{w,w'} D \setminus \{i\}^{\nabla} E_w$. Then, by Theorem 5, we have that $D \setminus \{i\}$ is decisive with respect to ∇ , a contradiction with respect to the assumption of minimality of D .

By the statement (16) and **(ESF1)**, it follows that $B(\nabla(\Phi, E_{w',w''})) \equiv \varphi_{w'}$. From this and the fact that $w \models B(\nabla(\Phi, E_{w,w'}))$, by (B-Rep) and the transitivity of \geq_{Φ} , we have that $B(\nabla(\Phi, E_{w,w''})) \equiv \varphi_w$. Therefore, we have $B(\nabla(\Phi, E_{w,w''})) \wedge B(E_{w''}) \vdash \perp$. By **(ESF-SD)**, there exists an N -profile Φ'' satisfying $B(\nabla(E''_i, E_{w,w''})) \wedge B(E_{w''}) \vdash \perp$ and $B(\nabla(E''_j, E_{w,w''})) \equiv B(E_{w''})$, for all j in $N \setminus \{i\}$. Hence, $B(\nabla(E_j, E_{w,w''})) \equiv B(\nabla(E''_j, E_{w,w''}))$, for all j in N . Then, by **(ESF-I)**, $B(\nabla(\Phi, E_{w,w''})) \equiv B(\nabla(\Phi'', E_{w,w''}))$. Thus, $B(\nabla(\Phi'', E_{w,w''})) \wedge B(E_{w''}) \vdash \perp$, and therefore, $E_{w,w''} \{i\}^{\nabla} E_{w''}$. Then, by Theorem 5, $\{i\}$ is decisive, contradicting again the minimality of D .

Now we know that D is decisive and has cardinality one. We conclude by Proposition 12. \square

Proposition 16. ∇^{π} is an ES basic fusion operator that satisfies **(ESF5)**, **(ESF7)**, **(ESF8W)**, **(ESF-SD)**, **(ESF-U)**, **(ESF-P)**, **(ESF-I)**, **(ESF-D)**, but does not satisfy **(ESF6)** and **(ESF8)**.

Proof. From the definition of $\Phi \mapsto \geq_{d_N}$, we have straightforwardly that it is a basic assignment which satisfies structure preserving and therefore, Property 1 holds. Now consider N a finite society of agents in \mathcal{S} , and let $\{N_1, N_2\}$ be a partition of N . Thus, since $\max(N) = \max\{\max(N_1), \max(N_2)\}$, if $w \geq_{d_{N_1}} w'$ and $w \geq_{d_{N_2}} w'$, then $w \geq_{d_N} w'$, that is, Property 3 holds. Similarly we prove that the projective assignment satisfies Property 4'.

To show that Property 2 does not hold, we consider a profile $\Phi = (\geq_1, \geq_2, \geq_3)$ such that $\llbracket B(\geq_1) \rrbracket = \llbracket B(\geq_2) \rrbracket = \{00\}$ and $\llbracket B(\geq_3) \rrbracket = \{00, 01\}$. Thus, $\llbracket \bigwedge_{i \in N} B(\geq_i) \rrbracket = \{00\}$, and therefore $\llbracket \bigwedge_{i \in N} B(\geq_i) \rrbracket \neq \llbracket B(\geq_3) \rrbracket$. To see that $\Phi \mapsto \geq_{d_N}$ does not satisfy Property 4, take the previous profile and consider $N_1 = \{1, 2\}$ and $N_2 = \{3\}$. Since $00 \succ_2 01$ and $00 \approx_3 01$, then $00 \succ_{\Phi \upharpoonright_{N_1}} 01$ and $00 \approx_{\Phi \upharpoonright_{N_2}} 01$, but $00 \approx_{\Phi} 01$.

Thus, from Proposition 2, it follows that ∇^{π} satisfies **(ESF5)**, **(ESF7)** and **(ESF8W)**, but **(ESF6)** and **(ESF8)** do not hold. Moreover, from Proposition 6, it follows that ∇^{π} satisfies **(ESF-P)**. Moreover, due to the freedom for building preorders over interpretations and the properties of projection and lexicographical combination, it is easy to see that ∇^{π} also satisfies **(ESF-SD)**, **(ESF-U)**, **(ESF-I)**. From this and Corollary 2, we have that it also satisfies **(ESF-D)**. \square

Proposition 17. Let \geq be a linear order over $\mathcal{W}_{\mathcal{P}}$. Then $\nabla^{\pi_{\geq}}$ is an ES fusion basic operator that satisfies **(ESF7)**, **(ESF8)**, **(ESF8W)**, **(ESF-U)**, **(ESF-P)**, **(ESF-I)** and **(ESF-D)**, but **(ESF5)**, **(ESF6)**, and **(ESF-SD)** do not hold.

Proof. First we will show that $\nabla^{\pi_{\geq}}$ satisfies **(ESF7)**, **(ESF8)** and **(ESF8W)**. In order to do this, by Proposition 2, it is enough to prove that $\Phi \mapsto \geq_{\Phi}^{\pi_{\geq}}$, the assignment associated to $\nabla^{\pi_{\geq}}$, satisfies the properties **3**, **4** and **4'** respectively.

To show that Property **3** holds, we note that, for all profile Φ , $\geq_{\Phi}^{\pi_{\geq}}$ is a linear order. Thus, consider N in $\mathcal{F}^*(S)$, $\{N_1, N_2\}$ a partition of N , let Φ be an N -profile and suppose w, w' is a pair of interpretations in \mathcal{W}_{Φ} such that $w \geq_{\Phi \upharpoonright N_1}^{\pi_{\geq}} w'$ and $w \geq_{\Phi \upharpoonright N_2}^{\pi_{\geq}} w'$. If $w = w'$ the result follows straightforwardly. Then, suppose that $w \neq w'$ and let us note that $w \geq_{\Phi \upharpoonright N_1}^{\pi_{\geq}} w'$. From this we have two cases: $w >_{d_{N_1}} w'$, or $w \simeq_{d_{N_1}} w'$ and $w > w'$. Thus, suppose that $w >_{d_{N_1}} w'$ (the case in which $w \simeq_{d_{N_1}} w'$ and $w > w'$ is similar). Since $w \geq_{\Phi \upharpoonright N_2}^{\pi_{\geq}} w'$, we also have either $w >_{d_{N_2}} w'$ or $w \simeq_{d_{N_2}} w'$ and $w > w'$. On the one hand, if $w >_{d_{N_2}} w'$, by virtue of $\max(N) = \max\{d_{N_1}, d_{N_2}\}$, we have $w >_{d_N} w'$. Therefore $w \geq_{\Phi}^{\pi_{\geq}} w'$. On the other hand, if $w \simeq_{d_{N_2}} w'$ and $w > w'$, we have that $w \geq_{d_N} w'$. From this and the fact that $w > w'$, it follows that $w \geq_{\Phi}^{\pi_{\geq}} w'$. Therefore, $\Phi \mapsto \geq_{\Phi}^{\pi_{\geq}}$ satisfies Property **3**. Similarly we obtain that Properties **4** and **4'** hold.

In order to show that a quasilinearized projective operator satisfies **(ESF-U)**, we will prove that $\Phi \mapsto \geq_{\Phi}^{\pi_{\geq}}$ satisfies property **(u)** given in Proposition 5. Thus, consider N in $\mathcal{F}^*(S)$, Φ an N -profile and \geq' an epistemic state such that $\geq_i = \geq'$, for all i in N . Thus, $\geq^{\text{lex}(\geq_{d_N}, \geq)} = \geq^{\text{lex}(\geq', \geq)}$, showing that $\geq_{\Phi}^{\pi_{\geq}} = \geq_{\geq'}^{\pi_{\geq}}$.

To see that **(ESF-I)** holds, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{\pi_{\geq}}$ satisfies property **(ind)** given in Proposition 8. Thus, consider N in $\mathcal{F}^*(S)$, Φ and Φ' a pair of N -profiles, and w, w' two interpretations such that $\geq_i \upharpoonright_{\{w, w'\}} = \geq'_i \upharpoonright_{\{w, w'\}}$, for all i in N . From the fact that $\geq_{d_N} \upharpoonright_{\{w, w'\}} = \geq'_{d_N} \upharpoonright_{\{w, w'\}}$, we have $\geq^{\text{lex}(\geq_{d_N}, \geq)} \upharpoonright_{\{w, w'\}} = \geq^{\text{lex}(\geq'_{d_N}, \geq)} \upharpoonright_{\{w, w'\}}$, that is, $\geq_{\Phi}^{\pi_{\geq}} \upharpoonright_{\{w, w'\}} = \geq_{\Phi'}^{\pi_{\geq}} \upharpoonright_{\{w, w'\}}$.

Now, since $\nabla^{\pi_{\geq}}$ satisfies Unanimity and Independence conditions, by Proposition 9 we have that **(ESF-P)** holds.

In order to show that $\nabla^{\pi_{\geq}}$ satisfies **(ESF-D)**, we will prove that its assignment satisfies Property **(d)** given in Proposition 10. Thus, assume N is in $\mathcal{F}^*(S)$, Φ is an N -profile and suppose that w, w' are interpretations in \mathcal{W}_{Φ} such that $w >_{d_N} w'$. Then $w \geq_{\Phi}^{\text{lex}(\geq_{d_N}, \geq)} w'$, that is $w \geq_{\Phi}^{\pi_{\geq}} w'$, as desired.

Now, given a profile Φ , since $\geq_{\Phi}^{\pi_{\geq}}$ is a linear order over \mathcal{W}_{Φ} , then, for every total preorder \geq over \mathcal{W}_{Φ} , we have that $\geq^{\text{lex}(\geq, \geq_{\Phi}^{\pi_{\geq}})}$ is also a linear order over \mathcal{W}_{Φ} , that is, $\nabla^{\pi_{\geq}}(\Phi, \geq)$ is a linear order over interpretations. From this it follows straightforwardly that $\nabla^{\pi_{\geq}}$ does not satisfy **(ESF-SD)**.

To show that $\nabla^{\pi_{\geq}}$ does not satisfy **(ESF5)** and **(ESF6)**, by virtue of Proposition 2, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{\pi_{\geq}}$ does not satisfy the Properties **1** and **2** respectively.

To show that Property **1** does not hold, consider w, w' a pair of interpretations in \mathcal{W}_{Φ} , with $w > w'$, and \geq_1, \geq_2 a pair of epistemic states satisfying that $w \simeq_1 w'$, $w >_2 w'$ and $\geq_1 \upharpoonright_{\mathcal{W}_{\Phi} \setminus \{w, w'\}} = \geq_2 \upharpoonright_{\mathcal{W}_{\Phi} \setminus \{w, w'\}}$. Thus $\geq_1 \neq \geq_2$, but $\geq^{\text{lex}(\geq_1, \geq)} = \geq^{\text{lex}(\geq_2, \geq)}$. That is, $\geq_{\geq_1}^{\pi_{\geq}} = \geq_{\geq_2}^{\pi_{\geq}}$.

Finally, to see that Property **2** does not hold, consider w, w' a pair of interpretations in \mathcal{W}_{Φ} such that $w > w'$, $N = \{1, 2\}$ and $\Phi = (\geq_1, \geq_2)$ an N -profile with $\max(\geq_i) = \{w, w'\}$, for $i = 1, 2$. Thus, $\llbracket \bigwedge_{i \in N} B(\geq_i) \rrbracket = \{w, w'\}$ and by definition we have $\max(\geq_{\Phi}^{\pi_{\geq}}) = \{w\}$. This shows that $\llbracket \bigwedge_{i \in N} B(\geq_i) \rrbracket \neq \max(\geq_{\Phi}^{\pi_{\geq}})$. \square

Proposition 18. *Let \geq be a linear order over \mathcal{W}_{Φ} . Then $\nabla^{\text{Q}\pi_{\geq}}$ is an ES fusion basic operator that satisfies **(ESF5)**, **(ESF8W)**, **(ESF-SD)**, **(ESF-P)**, **(ESF-I)** and **(ESF-D)**, but **(ESF6)**, **(ESF7)**, **(ESF8)**, and **(ESF-U)** do not hold.*

Proof. The proof that $\nabla^{\text{Q}\pi_{\geq}}$ satisfies **(ESF8W)**, **(ESF-I)** and **(ESF-D)** but not **(ESF6)**, is similar to that in Proposition 18. Now, in order to show that $\nabla^{\text{Q}\pi_{\geq}}$ satisfies **(ESF5)**, by Proposition 2, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{\text{Q}\pi_{\geq}}$, the assignment associated to this operator, satisfies **1**, but this follows straightforwardly from the fact that $\Phi \mapsto \geq_{\Phi}^{\text{Q}\pi_{\geq}}$ is structure preserving. Due to the freedom for building total preorders over \mathcal{W}_{Φ} and the definition of $\nabla^{\text{Q}\pi_{\geq}}$ it follows that it satisfies **(ESF-SD)**. Moreover, since $\nabla^{\text{Q}\pi_{\geq}}$ is a dictatorial operator, by Proposition 11, we have that **(ESF-P)** holds.

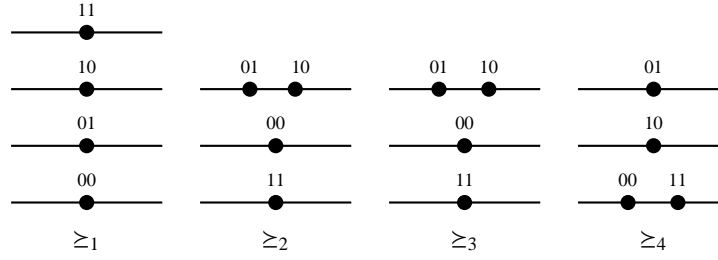


Fig. 1. The profile of epistemic states.

In order to prove that **(ESF7)** and **(ESF8)** do not hold, by Proposition 2, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{Q\pi_{\geq}}$ does not satisfy Properties **3** and **4** respectively. Consider the finite society $N = \{1, 2\}$, and its partition $N_1 = \{1\}$, $N_2 = \{2\}$. Let w, w' be a pair of interpretations in $\mathcal{W}_{\mathcal{P}}$ such that $w' > w$ and $\Phi = (\geq_1, \geq_2)$ be an N -profile such that $w \geq_1 w'$, $w \approx_2 w'$. From the definition of the assignment we have $w >_{\Phi \upharpoonright_{N_1}}^{Q\pi_{\geq}} w'$ and $w \geq_{\Phi \upharpoonright_{N_2}}^{Q\pi_{\geq}} w'$, but $w' >_{\Phi}^{Q\pi_{\geq}} w$.

Finally, to show that **(ESF-U)** does not hold, consider w, w' a pair of interpretations in $\mathcal{W}_{\mathcal{P}}$, such that $w > w'$. Let \geq^* be a total preorder over $\mathcal{W}_{\mathcal{P}}$ such that $\max(\geq^*) = \{w, w'\}$. Define $\Phi = (\geq_1, \geq_2)$ by putting $\geq_i = \geq^*$, for $i = 1, 2$. Thus, by definition $w >_{\Phi}^{Q\pi_{\geq}} w'$ and $\geq_{\geq^*}^{Q\pi_{\geq}} = \geq^*$ and, therefore, $\geq_{\Phi}^{Q\pi_{\geq}} \neq \geq_{\geq^*}^{Q\pi_{\geq}}$. The result follows from Proposition 5. \square

Proposition 19. $\nabla^{\Sigma-P\pi}$ is an ES basic fusion operator that satisfies **(ESF5)**, **(ESF6)**, **(ESF-SD)**, **(ESF-U)**, **(ESF-P)**, and **(ESF-D)**, but does not satisfy **(ESF7)**, **(ESF8)**, **(ESF8W)** nor **(ESF-I)**.

Proof. In order to see that $\nabla^{\Sigma-P\pi}$ satisfies **(ESF5)** and **(ESF6)**, by Proposition 2, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{\Sigma-P\pi}$ satisfies Properties **1** and **2**. Assume N in $\mathcal{F}^*(S)$ and consider an N -profile Φ . Let w and w' be a pair of interpretations. Since $\geq_{\geq}^{\Sigma} = \geq$ and $\geq^{\text{lex}(\geq, \geq)} = \geq$ we have that $\geq_{\geq}^{\Sigma-P\pi} = \geq$. Thus $\Phi \mapsto \geq_{\Phi}^{\Sigma-P\pi}$ is structure preserving. Therefore, it satisfies Property **1**.

In order to show that Property **2** holds, we first suppose that w is a model of $\bigwedge_{i \in N} B(\geq_i)$. Thus, $w \models B(\geq_{d_N})$ and by the maximality condition we have that $w \geq_{d_N} w'$. Since $\Phi \mapsto \geq_{\Phi}^{\Sigma}$ satisfies **2**, $w \geq_{\Phi}^{\Sigma} w'$. Thus, $w \geq_{\Phi}^{\Sigma-P\pi} w'$, that is, w is in $\max(\geq_{\Phi}^{\Sigma-P\pi})$. Suppose now, towards a contradiction, that w is in $\max(\geq_{\Phi}^{\Sigma-P\pi})$ and $w \not\models \bigwedge_{i \in N} B(\geq_i)$. Thus, if $w' \models \bigwedge_{i \in N} B(\geq_i)$, by **2** we have that $w' >_{\Phi}^{\Sigma} w$. On the other hand, since $w \geq_{\Phi}^{\Sigma-P\pi} w'$ it satisfies either $w >_{d_N} w'$ (in this case, by the maximality condition, $w' \not\models B(\geq_{d_N})$) or $w \approx_{d_N} w'$ and $w \geq_{\Phi}^{\Sigma} w'$. In both cases we get a contradiction. Thus Property **2** is proved.

In order to see that $\nabla^{\Sigma-P\pi}$ does not satisfy **(ESF7)**, **(ESF8)** and **(ESF8W)**, by Proposition 2, it is enough to see that $\Phi \mapsto \geq_{\Phi}^{\Sigma-P\pi}$ does not satisfy Properties **3**, **4** and **4'**. Consider the set of interpretations $\mathcal{W}_{\mathcal{P}} = \{00, 01, 10, 11\}$ of a finite propositional language $\mathcal{L}_{\mathcal{P}}$ with two propositional variables. Let $N = \{1, 2, 3, 4\}$. Let Φ be the N -profile represented in figure 1. Consider the partition of N given by $N_1 = \{1, 2\}$, $N_2 = \{3, 4\}$. Let us note the following:

- Since $00 >_2 11$, we have $00 >_{\Phi \upharpoonright_{N_1}}^{\Sigma-P\pi} 11$
- Since $00 >_3 11$ and $00 \approx_4 11$ then $00 >_{\Phi \upharpoonright_{N_2}}^{\Sigma} 11$. Hence $00 >_{\Phi}^{\Sigma-P\pi} 11$

However, since $11 \approx_4 00$ and easy calculations lead to $11 >_{\Phi}^{\Sigma} 00$, we have that $11 >_{\Phi}^{\Sigma-P\pi} 00$. Thus, **3**, **4** and **4'** do not hold.

Moreover, due to the freedom for building total preorders and because the basic assignment preserves the structure of epistemic states, we have that the Σ -pseudoprojective ES basic fusion operator satisfies **(ESF-SD)**.

In order to show that **(ESF-U)** holds, consider N in $\mathcal{F}^*(\mathcal{S})$, an N -profile Φ , \succeq an epistemic state such that $\succeq_i = \succeq$ for each i in N . Thus $\succeq_\Phi = \succeq'$, and therefore $\succeq_\Phi^{\Sigma-P\pi} = \succeq^{\text{lex}(\succeq, \succeq)} = \succeq$. From this, the result follows using Proposition 5.

Straightforward from the definition, we can see that $\nabla^{\Sigma-P\pi}$ has a dictator. Thus, **(ESF-D)** holds. Then, by Proposition 11, **(ESF-P)** holds.

We know that ∇^Σ does not satisfy **(ESF-I)**. Thus, by Proposition 8, there exists a society D , two D -profiles Φ and Φ' and two interpretations w and w' such that $\succeq_i \upharpoonright_{\{w, w'\}} = \succeq'_i \upharpoonright_{\{w, w'\}}$ for each $i \in D$, but $w \succ_\Phi^\Sigma w'$ and $w' \succeq_{\Phi'}^\Sigma w$. Choose an agent j such that $j > d_D$. Consider now the society $N = D \cup \{j\}$ and let \succeq_j be the flat preorder (all interpretations are indifferent). Let $\Phi_1 = \Phi \sqcup \succeq_j$ and $\Phi_2 = \Phi' \sqcup \succeq_j$. It is clear that $\succeq_i \upharpoonright_{\{w, w'\}} = \succeq'_i \upharpoonright_{\{w, w'\}}$ for each $i \in N$. We can also check that $\succeq_{\Phi_1}^{\Sigma-P\pi} = \succeq_\Phi^\Sigma$ and $\succeq_{\Phi_2}^{\Sigma-P\pi} = \succeq_{\Phi'}^\Sigma$. Therefore, $w \succ_{\Phi_1}^{\Sigma-P\pi} w'$ and $w' \succeq_{\Phi_2}^{\Sigma-P\pi} w$. That is, again by Proposition 8, **(ESF-I)** does not hold. \square